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New realization of cyclotomic q-Schur algebras I

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Abstract. We introduce a Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ and an associative algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ associated with the Cartan data of $\mathfrak{gl}_m$ which is separated into r parts with respect to $\mathbf{m} = (m_1, \dots, m_r)$ such that $m_1 + \cdots + m_r = m$. We show that the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ is a filtered deformation of the current Lie algebra of $\mathfrak{gl}_m$, and we can regard the algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a "q-analogue" of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$. Then, we realize a cyclotomic q-Schur algebra as a quotient algebra of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ under a certain mild condition. We also study the representation theory for $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ and $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$, and we apply them to the representations of the cyclotomic q-Schur algebras.

Contents

§ 0. Introduction

0.1. Let $\mathscr{H}_{n,r}$ be the Ariki-Koike algebra associated with the complex reflection group of type $G(r,1,n)$ over a commutative ring R with parameters $q, Q_0, \dots, Q_{r-1} \in R$, where q is invertible in R. Let $\mathscr{S}_{n,r}(\mathbf{m})$ be the cyclotomic q-Schur algebra associated with $\mathscr{H}_{n,r}$ introduced in [DJM], where $\mathbf{m} = (m_1, \dots, m_r)$ is an r-tuple of positive integers. By the result in [DJM], it is known that $\mathscr{S}_{n,r}(\mathbf{m})$-mod is a highest weight cover of $\mathscr{H}_{n,r}$-mod in the sense of [R] if R is a field and $\mathbf{m}$ is enough large.

In [RSVV] and [L] independently, it is proven that $\mathscr{S}_{n,r}(\mathbf{m})$-mod is equivalent to a certain highest weight subcategory of an affine parabolic category $\mathbf{O}$ in a dominant case of an affine general linear Lie algebra as a highest weight cover of $\mathscr{H}_{n,r}$-mod. It is also equivalent to the category $\mathcal{O}$ of rational Cherednik algebra with

the corresponding parameters. In the argument of [RSVV], the monoidal structure on the affine parabolic category $\mathbf{O}$ (more precisely, the structure of $\mathbf{O}$ as a bimodule category over the Kazhdan-Lusztig category) has an important role.

In the case where $r=1$, it is known that the q-Schur algbera $\mathscr{S}_{n,1}(m)$ is a quotient algebra of the quantum group $U_q(\mathfrak{gl}_m)$ associated with the general linear lie algebra $\mathfrak{gl}_m$, and $\bigoplus_{n\geq 0}\mathscr{S}_{n,1}(m)$-mod is equivalent to the category $\mathcal{C}^{\geq 0}_{U_q(\mathfrak{gl}_m)}$ consisting of finite dimensional polynomial representations of $U_q(\mathfrak{gl}_m)$ ([BLM], [D] and [J]). The category $\mathcal{C}^{\geq 0}_{U_q(\mathfrak{gl}_m)}$ has a (braided) monoidal structure which comes from the structure of $U_q(\mathfrak{gl}_m)$ as a Hopf algebra. Then the monoidal structure on $\mathcal{C}^{\geq 0}_{U_q(\mathfrak{gl}_m)}$ is compatible with the monoidal structure on the Kazhdan-Lusztig category by [KL]. However, it is not known such structures for cyclotomic q-Schur algebras in the case where $r>1$ although we may expect such structures through the equivalence in [RSVV]. This is a motivation of this paper.

In [W1], we obtained a presentation of cyclotomic q-Schur algebras by generators and defining relations. The argument in [W1] are based on the existence of the upper (resp. lower) Borel subalgebra of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}(\mathbf{m})$ which is introduced in [DR]. In [DR], it is proven that the upper (resp. lower) Borel subalgebra of $\mathscr{S}_{n,r}(\mathbf{m})$ is isomorphic to the upper (resp. lower) Borel subalgebra of $\mathscr{S}_{n,1}(m)$ (i.e. the case where $r=1$) which is a quotient of the upper (resp. lower) Borel subalgebra of the quantum group $U_q(\mathfrak{gl}_m)$ ($m:=\sum_{k=1}^r m_k$) if $\mathbf{m}$ is enough large. The presentation of $\mathscr{S}_{n,r}(\mathbf{m})$ in [W1] is applied to the representation theory of cyclotomic q-Schur algebras in [W2] and [W3]. However, this presentation is not so useful in general since, in the presentation, we need some non-commutative polynomials which are computable, but we can not describe them explicitly (see [W1, Lemma 7.2]). Hence, we hope more useful realization of cyclotomic q-Schur algebras like as the fact that the q-Schur algebra $\mathscr{S}_{n,1}(m)$ is a quotient of the quantum group $U_q(\mathfrak{gl}_m)$ in the case where $r=1$. In this paper, by extending the argument in [W1], we give a possibility of such realization of cyclotomic q-Schur algebras.

0.2. Let $\mathbf{Q}=(Q_1,\dots,Q_{r-1})$ be an $r-1$ tuple of indeterminate elements over $\mathbb{Z}$, and $\mathbb{Q}(\mathbf{Q})$ be a field of rational functions with variables $\mathbf{Q}$. In §2, we introduce a Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ with parameters $\mathbf{Q}$ associated with the Cartan data of $\mathfrak{gl}_m$ ($m=\sum_{k=1}^r m_k$) which is separated into r parts with respect to $\mathbf{m}$ (see the paragraph 1.3). Then, in Proposition 2.13, we prove that $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ is a filtered deformation of the current Lie algebra $\mathfrak{gl}_m[x]=\mathbb{Q}(\mathbf{Q})[x]\otimes\mathfrak{gl}_m$ of the general linear Lie algebra $\mathfrak{gl}_m$.

In Corollary 2.8, we see that $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ has a triangular decomposition

$$\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})=\mathfrak{n}^-\oplus\mathfrak{n}^0\oplus\mathfrak{n}^+.$$

Then we can develop the weight theory to study representations of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ in the usual manner (see §3). Let $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ be the category of finite dimensional $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$-modules which have the weight space decompositions, and all eigenvalues of the action of $\mathfrak{n}^0$ belong to $\mathbb{Q}(\mathbf{Q})$. Then we see that a simple $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$-module in $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ is a highest weight module.

There exists a surjective homomorphism of Lie algebras $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})\to\mathfrak{gl}_m$ (see (2.16.1)) which can be regarded as a special case of evaluation homomorphisms (see

Remark 2.17). Let $\mathcal{C}_{\mathfrak{gl}_m}$ be the category of finite dimensional $\mathfrak{gl}_m$-modules which have the weight space decompositions. Then $\mathcal{C}_{\mathfrak{gl}_m}$ is a full subcategory of $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ through the above surjection (see Proposition 3.7).

Let $\widetilde{\mathbf{Q}} = (Q_0, Q_1, \dots, Q_{r-1})$ be an r tuple of indeterminate elements over $\mathbb{Z}$, and $\mathbb{Q}(\widetilde{\mathbf{Q}})$ be a field of rational functions with variables $\widetilde{\mathbf{Q}}$. Put $\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}) = \mathbb{Q}(\widetilde{\mathbf{Q}}) \otimes_{\mathbb{Q}(\mathbf{Q})} \mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$, and define the category $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ in a similar way. Let $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ be the cyclotomic q-Schur algebra over $\mathbb{Q}(\widetilde{\mathbf{Q}})$ with parameters $q = 1$ and $\widetilde{\mathbf{Q}}$. In Theorem 8.4, we prove that there exists a homomorphism of algebras

$$\Psi_{\mathbf{1}} : U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})) \to \mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m}),$$

where $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$ is the universal enveloping algebra of $\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$. Assume that $m_k \geq n$ for all $k = 1, 2, \dots, r-1$, then $\Psi_{\mathbf{1}}$ is surjective. Then $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod is a full subcategory of $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the surjection $\Psi_{\mathbf{1}}$ (see Theorem 8.4 (ii)). We expect that the surjectivity of $\Psi_{\mathbf{1}}$ also holds without the condition for $\mathbf{m}$. (We need the condition for $\mathbf{m}$ by a technical reason (see Remark 8.2).)

It is known that $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ is semi-simple, and the set of Weyl (cell) modules $\{\Delta(\lambda) \mid \lambda \in \widetilde{\varLambda}^+_{n,r}(\mathbf{m})\}$ gives a complete set of isomorphism classes of simple $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules (see §6 and [DJM] for definitions). The characters of the Weyl modules, denoted by $\operatorname{ch}\Delta(\lambda)$ ($\lambda \in \widetilde{\varLambda}^+_{n,r}(\mathbf{m})$), are studied in [W2]. We see that $\operatorname{ch}\Delta(\lambda)$ ($\lambda \in \widetilde{\varLambda}^+_{n,r}(\mathbf{m})$) is a symmetric polynomial with variables $\mathbf{x}_{\mathbf{m}}$ with respect to $\mathbf{m}$. Put $\widetilde{\varLambda}^+_{\geq 0}(\mathbf{m}) = \cup_{n \geq 0} \widetilde{\varLambda}^+_{n,r}(\mathbf{m})$. Then, for $\lambda, \mu \in \widetilde{\varLambda}^+_{\geq 0}(\mathbf{m})$, it was conjectured that

$$\operatorname{ch}\Delta(\lambda)\operatorname{ch}\Delta(\mu) = \sum_{\nu \in \widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})} \mathrm{LR}^{\nu}_{\lambda\mu} \operatorname{ch}\Delta(\nu) \tag{0.2.1}$$

in [W2], where $\mathrm{LR}^{\nu}_{\lambda\mu}$ is the product of Littlewood-Richardson coefficients with respect to λ, μ and ν (see §9 for details). We prove this conjecture in Proposition 9.4. We remark that the characters of Weyl modules of a cyclotomic q-Schur algebra do not depend on the choice of a base field and parameters.

By using the usual coproduct of the universal enveloping algebra $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$ of $\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$, we can consider the tensor product $M \otimes N$ in $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-mod for $M, N \in U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-mod. We regard $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules ($n \geq 0$) as a $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules through the homomorphism $\Psi_{\mathbf{1}}$. Take $n, n_1, n_2 \in \mathbb{Z}_{>0}$ such that $n = n_1 + n_2$. Then, in Proposition 10.1, we prove that, for $\lambda \in \widetilde{\varLambda}^+_{n_1,r}(\mathbf{m})$ and $\mu \in \widetilde{\varLambda}^+_{n_2,r}(\mathbf{m})$,

$$\Delta(\lambda) \otimes \Delta(\mu) \cong \bigoplus_{\nu \in \widetilde{\varLambda}^+_{n,r}(\mathbf{m})} \mathrm{LR}^{\nu}_{\lambda\mu} \Delta(\nu) \tag{0.2.2}$$

as $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules if $m_k \geq n$ for all $k = 1, 2, \dots, r-1$, where $\mathrm{LR}^{\nu}_{\lambda\mu}\Delta(\nu)$ means the direct sum of $\mathrm{LR}^{\nu}_{\lambda\mu}$ copies of $\Delta(\nu)$. In particular, we see that $\Delta(\lambda) \otimes \Delta(\nu) \in \mathscr{S}_{n,r}(\mathbf{m})$-mod. The decomposition (0.2.2) gives an interpretation of the formula

(0.2.1) in the category $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$. We expect that (0.2.2) also holds without the condition for $\mathbf{m}$. (Note that we prove the formula (0.2.1) without the condition for $\mathbf{m}$ in Proposition 9.4.)

0.3. Put $\mathbb{A} = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_{r-1}]$, where $q, Q_1, \dots, Q_{r-1}$ are indeterminate elements over $\mathbb{Z}$, and let $\mathbb{K} = \mathbb{Q}(q, Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{A}$. In §4, we introduce an associative algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ with parameters q and $\mathbf{Q}$ associated with the Cartan data of $\mathfrak{gl}_m$ which is separated into r parts with respect to $\mathbf{m}$.

Let $\mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-subalgebra of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ generated by defining generators of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ (see the paragraph 4.11). We regard $\mathbb{Q}(\mathbf{Q})$ as an $\mathbb{A}$-module through the ring homomorphism $\mathbb{A} \to \mathbb{Q}(\mathbf{Q})$ by sending q to 1, and we consider the specialization $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{Q},q,\mathbf{Q}}(\mathbf{m})$ using this ring homomorphism. Then we have a surjective homomorphism of algebras

$$U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})) \to \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})/\mathfrak{J}, \tag{0.3.1}$$

where $\mathfrak{J}$ is a certain ideal of $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ (see (4.11.2)). We conjecture that the surjection (0.3.1) is isomorphic. Then we can regard $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a "q-analogue" of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$. Dividing by the ideal $\mathfrak{J}$ in (0.3.1) means that the Cartan subalgebra $U(\mathfrak{n}^0)$ of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ deforms to several directions in $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ (see the paragraph 4.11 and Remark 4.12).

We see that $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ has a triangular decomposition

$$\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) = \mathcal{U}^{-}\mathcal{U}^{0}\mathcal{U}^{+} \tag{0.3.2}$$

in a weak sense (see (4.6.1)). We conjecture that the multiplication map $\mathcal{U}^{-} \otimes_{\mathbb{K}} \mathcal{U}^{0} \otimes_{\mathbb{K}} \mathcal{U}^{+} \to \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ gives an isomorphism as vector spaces. More precisely, we expect the existence of a PBW type basis of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ which is compatible with a PBW basis of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ through the homomorphism (0.3.1).

Anyway, thanks to the triangular decomposition (0.3.2), we can develop the weight theory to study $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-modules in the usual manner (see §5). Let $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ be the category of finite dimensional $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-modules which have the weight space decompositions, and all eigenvalues of the action of $\mathcal{U}^0$ belong to $\mathbb{K}$. Then we see that a simple $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-module in $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ is a highest weight module.

There exists a surjective homomorphism of algebras $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \to U_q(\mathfrak{gl}_m)$ (see (4.9.1)) which can be regarded as a special case of evaluation homomorphisms (see Remark 4.10). Let $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ be the category of finite dimensional $U_q(\mathfrak{gl}_m)$-modules which have the weight space decompositions. Then $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ is a full subcategory of $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ through the above surjection (see Proposition 5.6).

Put $\widetilde{\mathbb{K}} = \mathbb{K}(Q_0)$ and $\widetilde{\mathbb{A}} = \mathbb{A}[Q_0]$. We also put $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. Let $\mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-form of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ taking divided powers (see the paragraph 4.13), and put $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{A}} \otimes_{\mathbb{A}} \mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$. Let $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ (resp. $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$) be the cyclotomic q-Schur algebra over $\widetilde{\mathbb{K}}$ (resp. over $\widetilde{\mathbb{A}}$) with parameters q and $\widetilde{\mathbf{Q}}$. In

Theorem 8.1, we prove that there exists a homomorphism of algebras

$$\Psi : \mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m}).$$

By the restriction of Ψ to $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m})$, we have the homomorphism $\Psi_{\widetilde{\mathbb{A}}} : \mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$. Then we can specialize $\Psi_{\widetilde{\mathbb{A}}}$ to any base ring and parameters. If $m_k \geq n$ for all $k = 1, 2, \dots, r-1$, then Ψ (resp. $\Psi_{\widetilde{\mathbb{A}}}$) is surjective (see also Remark 8.2 for surjectivity of Ψ). In Theorem 8.3, we prove that $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$-mod is a full subcategory of $\mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the surjection Ψ if $\mathbf{m}$ is enough large.

We conjecture that $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ has a structure as a Hopf algebra, and that the decomposition (0.2.2) also holds for Weyl modules of $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ ($n \geq 0$) through the homomorphism Ψ and the Hopf algebra structure of $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$. (Note that the formula (0.2.1) holds for $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ ($n \geq 0$).)

It is also interesting problem to obtain a monoidal structure for $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ and its specialization) which should be related to the monoidal structure on the affine parabolic category $\mathbf{O}$.

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§ 1. Notation

1.1. For a condition X, put $\delta_{(X)} = \begin{cases} 1 & \text{if } X \text{ is true,} \\ 0 & \text{if } X \text{ is false.} \end{cases}$ We also put $\delta_{i,j} = \delta_{(i=j)}$ for simplicity.

1.2. ***q*-integers.** Let $\mathbb{Q}(q)$ be the field of rational functions over $\mathbb{Q}$ with an indeterminate variable q. For $d \in \mathbb{Z}$, put $[d] = (q^d - q^{-d})/(q - q^{-1}) \in \mathbb{Q}(q)$. For $d \in \mathbb{Z}_{>0}$, put $[d]! = [d][d-1]\dots[1]$, and we put $[0]! = 1$. For $d \in \mathbb{Z}$ and $c \in \mathbb{Z}_{>0}$, put

$$\begin{bmatrix} d \\ c \end{bmatrix} = \frac{[d][d-1]\dots[d-c+1]}{[c][c-1]\dots[1]}, \text{ and put } \begin{bmatrix} d \\ 0 \end{bmatrix} = 1.$$

It is well-known that all $[d], [d]!$ and $\left[\begin{smallmatrix} d \\ c \end{smallmatrix}\right]$ belong to $\mathbb{Z}[q, q^{-1}]$. Thus we can specialize these elements to any ring R and $q \in R$ such that q is invertible in R, and we denote them by same symbols.

1.3. Cartan data. Let $\mathbf{m} = (m_1, \dots, m_r)$ be an r-tuple of positive integers. Put $m = \sum_{k=1}^r m_k$. Let $P = \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ be the weight lattice of $\mathfrak{gl}_m$, and let $P^\vee = \bigoplus_{i=1}^m \mathbb{Z}h_i$ be its dual with the natural pairing $\langle\, ,\, \rangle : P \times P^\vee \to \mathbb{Z}$ such that $\langle \varepsilon_i, h_j \rangle = \delta_{ij}$. put $P_{\geq 0} = \bigoplus_{i=1}^m \mathbb{Z}_{\geq 0}\varepsilon_i$.

Set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, m-1$, then $\Pi = \{\alpha_i \mid 1 \leq i \leq m-1\}$ is the set of simple roots, and $Q = \bigoplus_{i=1}^{m-1} \mathbb{Z}\alpha_i$ is the root lattice of $\mathfrak{gl}_m$. Put $Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{\geq 0}\alpha_i$.

Set $\alpha_i^\vee = h_i - h_{i+1}$ for $i = 1, \dots, m-1$, then $\Pi^\vee = \{\alpha_i^\vee \mid 1 \leq i \leq m-1\}$ is the set of simple coroots.

We define a partial order $\geq$ on P, so called dominance order, by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Put $\varGamma(\mathbf{m}) = \{(i,k) \,|\, 1 \leq i \leq m_k, 1 \leq k \leq r\}$, and $\varGamma'(\mathbf{m}) = \varGamma(\mathbf{m}) \setminus \{(m_r, r)\}$. We identify the set $\varGamma(\mathbf{m})$ with the set $\{1, 2, \dots, m\}$ by the bijection

$$\gamma : \varGamma(\mathbf{m}) \to \{1, 2, \dots, m\} \text{ such that } (i,k) \mapsto \sum_{j=1}^{k-1} m_j + i. \tag{1.3.1}$$

Then, we can identify the set $\varGamma'(\mathbf{m})$ with the set $\{1, 2, , \dots, m-1\}$. Under the identification (1.3.1), for $(i,k), (j,l) \in \varGamma(\mathbf{m})$, we define

$$(i,k) > (j,l) \text{ if } \gamma((i,k)) > \gamma((j,l)), \text{ and } (i,k) \pm (j,l) = \gamma((i,k)) \pm \gamma((j,l)).$$

We also have $(m_k + 1, k) = (1, k+1)$ for $k = 1, \dots, r-1$ (resp. $(1-1, k) = (m_{k-1}, k-1)$ for $k = 2, \dots, r$).

We may write

$$P = \bigoplus_{(i,k) \in \varGamma(\mathbf{m})} \mathbb{Z}\varepsilon_{(i,k)}, \quad P^\vee = \bigoplus_{(i,k) \in \varGamma(\mathbf{m})} \mathbb{Z}h_{(i,k)}, \quad Q = \bigoplus_{(i,k) \in \varGamma'(\mathbf{m})} \mathbb{Z}\alpha_{(i,k)}.$$

For $(i,k) \in \varGamma'(\mathbf{m})$, $(j,l) \in \varGamma(\mathbf{m})$, put $a_{(i,k)(j,l)} = \langle \alpha_{(i,k)}, h_{(j,l)} \rangle$. Then, we have

$$a_{(i,k)(j,l)} = \begin{cases} 1 & \text{if } (j,l) = (i,k), \\ -1 & \text{if } (j,l) = (i+1,k), \\ 0 & \text{otherwise.} \end{cases}$$

Put

$$P^+ = \{\lambda \in P \,|\, \langle \lambda, \alpha^\vee_{(i,k)} \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } (i,k) \in \varGamma'(\mathbf{m})\} \text{ and}$$
$$P^+_{\mathbf{m}} = \{\lambda \in P \,|\, \langle \lambda, \alpha^\vee_{(i,k)} \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } (i,k) \in \varGamma(\mathbf{m}) \setminus \{(m_k, k) \,|\, 1 \leq k \leq r\}\}.$$

Then P^+ is the set of dominant integral weights for $\mathfrak{gl}_m$, and $P^+_{\mathbf{m}}$ is the set of dominant integral weights for Levi subalgebra $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ of $\mathfrak{gl}_m$ with respect to $\mathbf{m} = (m_1, \dots, m_r)$.

§ 2. Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$

In this section, we introduce a Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ with $r-1$ parameters $\mathbf{Q} = (Q_1, \dots, Q_{r-1})$ associated with the Cartan data in the paragraph 1.3. Then we study some basic structures of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$. In particular, we prove that $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ is a filtered deformation of the current Lie algebra $\mathfrak{gl}_m[x]$ of the general linear Lie algebra $\mathfrak{gl}_m$.

2.1. Let $\mathbf{Q} = (Q_1, \dots, Q_{r-1})$ be an $r-1$-tuple of indeterminate elements over $\mathbb{Z}$. Let $\mathbb{Z}[\mathbf{Q}] = \mathbb{Z}[Q_1, \dots, Q_{r-1}]$ be the polynomial ring with variables $Q_1, \dots, Q_{r-1}$, and $\mathbb{Q}(\mathbf{Q}) = \mathbb{Q}(Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{Z}[\mathbf{Q}]$.

Definition 2.2. *We define the Lie algebra $\mathfrak{g} = \mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ over $\mathbb{Q}(\mathbf{Q})$ by the following generators and defining relations:*

Generators: $\mathcal{X}^{\pm}_{(i,k),t}$, $\mathcal{I}_{(j,l),t}$ $((i,k) \in \Gamma'(\mathbf{m}),\ (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0)$.

Relations:

$$\text{(L1)} \qquad [\mathcal{I}_{(i,k),s}, \mathcal{I}_{(j,l),t}] = 0,$$

$$\text{(L2)} \qquad [\mathcal{I}_{(j,l),s}, \mathcal{X}^{\pm}_{(i,k),t}] = \pm a_{(i,k)(j,l)} \mathcal{X}^{\pm}_{(i,k),s+t},$$

$$\text{(L3)} \qquad [\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{-}_{(j,l),s}] = \delta_{(i,k),(j,l)} \begin{cases} \mathcal{J}_{(i,k),s+t} & \text{if } i \neq m_k, \\ -Q_k \mathcal{J}_{(m_k,k),s+t} + \mathcal{J}_{(m_k,k),s+t+1} & \text{if } i = m_k, \end{cases}$$

$$\text{(L4)} \qquad [\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{X}^{\pm}_{(j,l),s}] = 0 \quad \text{if } (j,l) \neq (i \pm 1, k),$$

$$\text{(L5)} \qquad [\mathcal{X}^{+}_{(i,k),t+1}, \mathcal{X}^{+}_{(i\pm1,k),s}] = [\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{+}_{(i\pm1,k),s+1}],$$
$$[\mathcal{X}^{-}_{(i,k),t+1}, \mathcal{X}^{-}_{(i\pm1,k),s}] = [\mathcal{X}^{-}_{(i,k),t}, \mathcal{X}^{-}_{(i\pm1,k),s+1}],$$

$$\text{(L6)} \qquad [\mathcal{X}^{+}_{(i,k),s}, [\mathcal{X}^{+}_{(i,k),t}, \mathcal{X}^{+}_{(i\pm1,k),u}]] = [\mathcal{X}^{-}_{(i,k),s}, [\mathcal{X}^{-}_{(i,k),t}, \mathcal{X}^{-}_{(i\pm1,k),u}]] = 0,$$

where we put $\mathcal{J}_{(i,k),t} = \mathcal{I}_{(i,k),t} - \mathcal{I}_{(i+1,k),t}$.

2.3. For $\tau \in \mathbb{Q}(\mathbf{Q})$, let $V_\tau = \bigoplus_{(j,l)\in\Gamma(\mathbf{m})} \mathbb{Q}(\mathbf{Q}) v_{(j,l)}$ be the $\mathbb{Q}(\mathbf{Q})$-vector space with a basis $\{v_{(j,l)} \mid (j,l) \in \Gamma(\mathbf{m})\}$. We can define the action of $\mathfrak{g}$ on V_τ by

$$\mathcal{X}^{+}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(i,k)} & \text{if } (j,l) = (i+1,k) \text{ and } i \neq m_k, \\ (-Q_k + \tau)\tau^t v_{(m_k,k)} & \text{if } (j,l) = (1,k+1) \text{ and } i = m_k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{X}^{-}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(i+1,k)} & \text{if } (j,l) = (i,k), \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{I}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(j,l)} & \text{if } (j,l) = (i,k), \\ 0 & \text{otherwise.} \end{cases}$$

We can check the well-definedness of the above action by direct calculations.

2.4. For $(i,k), (j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$, we define the element $\mathcal{E}^t_{(i,k)(j,l)} \in \mathfrak{g}$ by

$$\mathcal{E}^t_{(i,k),(j,l)} = \begin{cases} \mathcal{I}_{(i,k),t} & \text{if } (j,l) = (i,k), \\ [\mathcal{X}^{+}_{(i,k),0}, [\mathcal{X}^{+}_{(i+1,k),0}, \dots, [\mathcal{X}^{+}_{(j-2,l),0}, \mathcal{X}^{+}_{(j-1,l),t}] \dots] & \text{if } (j,l) > (i,k), \\ [\mathcal{X}^{-}_{(i-1,k),0}, [\mathcal{X}^{-}_{(i-2,k),0}, \dots, [\mathcal{X}^{-}_{(j+1,l),0}, \mathcal{X}^{-}_{(j,l),t}] \dots] & \text{if } (j,l) < (i,k), \end{cases}$$

in particular, we have $\mathcal{E}^t_{(i,k),(i+1,k)} = \mathcal{X}^{+}_{(i,k),t}$ and $\mathcal{E}^t_{(i+1,k),(i,k)} = \mathcal{X}^{-}_{(i,k),t}$.
If $(j,l) > (i,k)$, we have

$$\begin{aligned} \mathcal{E}^t_{(i,k),(j,l)} &= [\mathcal{X}^{+}_{(i,k),0}, \mathcal{E}^t_{(i+1,k),(j,l)}] \\ &= [\mathcal{E}^t_{(i,k),(j-1,l)}, \mathcal{X}^{+}_{(j-1,l),0}]. \end{aligned}$$

If $(j,l) < (i,k)$, we have

$$\begin{aligned}\mathcal{E}^t_{(i,k),(j,l)} &= [\mathcal{X}^-_{(i-1,k),0}, \mathcal{E}^t_{(i-1,k),(j,l)}] \\ &= [\mathcal{E}^t_{(i,k),(j+1,l)}, \mathcal{X}^-_{(j,l),0}].\end{aligned}$$

Lemma 2.5.

(i) *For $(i,k),(j,l) \in \Gamma(\mathbf{m})$ such that $(j,l) > (i,k)$, we have*

$$\text{(2.5.1)} \qquad [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i-1,k),(j,l)} & \text{if } (a,c) = (i-1,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j+1,l)} & \text{if } (a,c) = (j,l), \\ 0 & \text{otherwise}, \end{cases}$$

$$\text{(2.5.2)} \qquad [\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c) = (i,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c) = (j,l), \\ 0 & \text{otherwise}, \end{cases}$$

(2.5.3)

$$\begin{aligned}&[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] \\ &= \begin{cases} -\mathcal{E}^{t+s}_{(i,k),(i,k)} + \mathcal{E}^{t+s}_{(i+1,k),(i+1,k)} & \text{if } \ell = 1, (a,c) = (i,k) \text{ and } i \neq m_k, \\ Q_k(\mathcal{E}^{t+s}_{(m_k,k),(m_k,k)} - \mathcal{E}^{t+s}_{(1,k+1),(1,k+1)}) - \mathcal{E}^{t+s+1}_{(m_k,k),(m_k,k)} + \mathcal{E}^{t+s+1}_{(1,k+1),(1,k+1)} & \\ & \text{if } \ell = 1, (a,c) = (i,k) \text{ and } i = m_k, \\ \mathcal{E}^{t+s}_{(i+1,k),(j,l)} & \text{if } \ell > 1, (a,c) = (i,k) \text{ and } i \neq m_k, \\ -Q_k\mathcal{E}^{t+s}_{(1,k+1),(j,l)} + \mathcal{E}^{t+s+1}_{(1,k+1),(j,l)} & \text{if } \ell > 1, (a,c) = (i,k) \text{ and } i = m_k, \\ -\mathcal{E}^{t+s}_{(i,k),(j-1,l)} & \text{if } \ell > 1, (a,c) = (j-1,l) \text{ and } j-1 \neq m_l, \\ Q_l\mathcal{E}^{t+s}_{(i,k),(m_l,l)} - \mathcal{E}^{t+s+1}_{(i,k),(m_l,l)} & \text{if } \ell > 1, (a,c) = (j-1,l) \text{ and } j-1 = m_l, \\ 0 & \text{otherwise}, \end{cases}\end{aligned}$$

where we put $\ell = (j,l) - (i,k)$.

(ii) *For $(i,k),(j,l) \in \Gamma(\mathbf{m})$ such that $(j,l) < (i,k)$, we have*

$$[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i+1,k),(j,l)} & \text{if } (a,c) = (i,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j-1,l)} & \text{if } (a,c) = (j-1,l), \\ 0 & \text{otherwise}, \end{cases}$$

$$[\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c) = (i,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j,l)} & \text{if } (a,c) = (j,l), \\ 0 & \text{otherwise}, \end{cases}$$

$$[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}]$$

$$
= \begin{cases}
\mathcal{E}^{t+s}_{(i-1,k),(i-1,k)} - \mathcal{E}^{t+s}_{(i,k),(i,k)} & \text{\it if } \ell = 1, (a,c) = (i-1,k) \text{ \it and } i-1 \neq m_k, \\
-Q_k(\mathcal{E}^{t+s}_{(m_k,k),(m_k,k)} - \mathcal{E}^{t+s}_{(1,k+1),(1,k+1)}) + \mathcal{E}^{t+s+1}_{(m_k,k),(m_k,k)} - \mathcal{E}^{t+s+1}_{(1,k+1),(1,k+1)} & \\
 & \text{\it if } \ell = 1, (a,c) = (i-1,k) \text{ \it and } i-1 = m_k, \\
\mathcal{E}^{t+s}_{(i-1,k),(j,l)} & \text{\it if } \ell > 1, (a,c) = (i-1,k) \text{ \it and } i-1 \neq m_k, \\
-Q_k \mathcal{E}^{t+s}_{(m_k,k),(j,l)} + \mathcal{E}^{t+s+1}_{(m_k,k),(j,l)} & \text{\it if } \ell > 1, (a,c) = (i-1,k) \text{ \it and } i-1 = m_k, \\
-\mathcal{E}^{t+s}_{(i,k),(j+1,l)} & \text{\it if } \ell > 1, (a,c) = (j,l) \text{ \it and } j \neq m_l, \\
Q_l \mathcal{E}^{t+s}_{(i,k),(1,l+1)} - \mathcal{E}^{t+s+1}_{(i,k),(1,l+1)} & \text{\it if } \ell > 1, (a,c) = (j,l) \text{ \it and } j = m_l, \\
0 & \text{\it otherwise},
\end{cases}
$$

where we put $\ell = (i,k) - (j,l)$.

(iii) *For* $(i,k) \in \Gamma(\mathbf{m})$, *we have*

$$
\begin{aligned}
&[\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(i,k)}] = 0, \\
&[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(i,k)}] = -a_{(a,c)(i,k)} \mathcal{E}^{t+s}_{(a,c),(a+1,c)}, \\
&[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(i,k)}] = a_{(a,c)(i,k)} \mathcal{E}^{t+s}_{(a+1,c),(a,c)}.
\end{aligned}
$$

Proof. We prove (2.5.1) by the induction on $(j,l) - (i,k)$.

In the case where $(j,l)-(i,k) = 1$, it is follows from the relations (L4) and (L5). Assume that $(j,l) - (i,k) > 1$. We have

$$
\begin{aligned}
[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] &= [\mathcal{X}^+_{(a,c),s}, [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^t_{(i+1,k),(j,l)}]] \\
&= [\mathcal{X}^+_{(i,k),0}, [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i+1,k),(j,l)}]] + [[\mathcal{X}^+_{(a,c),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}].
\end{aligned}
$$

Applying the assumption of the induction, we have

$$
(2.5.4) \quad [\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases}
[\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i,k),(j,l)}] & \text{if } (a,c) = (i,k), \\
-[\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i+1,k),(j+1,l)}] & \text{if } (a,c) = (j,l), \\
[[\mathcal{X}^+_{(i-1,k),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i-1,k), \\
[[\mathcal{X}^+_{(i+1,k),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i+1,k), \\
0 & \text{otherwise.}
\end{cases}
$$

We also have

$$
\begin{aligned}
[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] &= [\mathcal{X}^+_{(a,c),s}, [\mathcal{E}^t_{(i,k),(j-1,l)}, \mathcal{X}^+_{(j-1,l),0}]] \\
&= [[\mathcal{X}^+_{(j-1,l),0}, \mathcal{X}^+_{(a,c),s}], \mathcal{E}^t_{(i,k),(j-1,l)}] + [[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j-1,l)}], \mathcal{X}^+_{(j-1,l),0}].
\end{aligned}
$$

Applying the assumption of the induction, we have

(2.5.5)
$$[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} [[\mathcal{X}^+_{(j-1,l),0}, \mathcal{X}^+_{(j,l),s}], \mathcal{E}^t_{(i,k),(j-1,l)}] & \text{if } (a,c) = (j,l), \\ [[\mathcal{X}^+_{(j-1,l),0}, \mathcal{X}^+_{(j-2,l),s}], \mathcal{E}^t_{(i,k),(j-1,l)}] & \text{if } (a,c) = (j-2,l), \\ [\mathcal{E}^{t+s}_{(i-1,k),(j-1,l)}, \mathcal{X}^+_{(j-1,l),0}] & \text{if } (a,c) = (i-1,k), \\ -[\mathcal{E}^{t+s}_{(i,k),(j,l)}, \mathcal{X}^+_{(j-1,l),0}] & \text{if } (a,c) = (j-1,l), \\ 0 & \text{otherwise.} \end{cases}$$

By (2.5.4) and (2.5.5), we have

$$[\mathcal{X}^+_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} \mathcal{E}^{t+s}_{(i-1,k),(j,l)} & \text{if } (a,c) = (i-1,k), \\ -\mathcal{E}^{t+s}_{(i,k),(j+1,l)} & \text{if } (a,c) = (j,l), \\ [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] & \text{if } (a,c) = (i,k) = (j-2,l), \\ [[\mathcal{X}^+_{(i+1,k),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(i+3,k)}] & \\ & \text{if } (a,c) = (i+1,k) = (j-2,l), \\ [\mathcal{X}^-_{(i+1,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] & \text{if } (a,c) = (i+1,k) = (j-1,l), \\ 0 & \text{otherwise.} \end{cases}$$

By the direct calculations using the relations (L4)-(L6), we also have

$$[\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] = [[\mathcal{X}^+_{(i+1,k),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(i+3,k)}] = [\mathcal{X}^-_{(i+1,k),0}, \mathcal{E}^{t+s}_{(i,k),(i+2,k)}] = 0.$$

Now we proved (2.5.1).

We prove (2.5.2) by the induction on $(j,l)-(i,k)$. In the case where $(j,l)-(i,k) = 1$, it is just the relation (L2). Assume that $(j,l)-(i,k) > 1$. We have

$$\begin{aligned} [\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] &= [\mathcal{I}_{(a,c),s}, [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^t_{(i+1,k),(j,l)}]] \\ &= [\mathcal{X}^+_{(i,k),0}, [\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i+1,k),(j,l)}]] + [[\mathcal{I}_{(a,c),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}]. \end{aligned}$$

Applying the assumption of the induction, we have

$$[\mathcal{I}_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = \begin{cases} [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i+1,k),(j,l)}] - [\mathcal{X}^+_{(i,k),s}, \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i+1,k), \\ -[\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i+1,k),(j,l)}] & \text{if } (a,c) = (j,l), \\ [\mathcal{X}^+_{(i,k),s}, \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i,k), \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have (2.5.2) by applying (2.5.1).

We prove (2.5.3) by the induction on $\ell = (j,l)-(i,k)$. In the case where $\ell = 1, 2$, we can show (2.5.3) by direct calculations. Assume that $\ell > 2$, we have

$$[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}] = [\mathcal{X}^-_{(a,c),s}, [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^t_{(i+1,k),(j,l)}]]$$

$$= [\mathcal{X}^+_{(i,k),0}, [\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i+1,k),(j,l)}]] + [[\mathcal{X}^-_{(a,c),s}, \mathcal{X}^+_{(i,k),0}], \mathcal{E}^t_{(i+1,k),(j,l)}].$$

Applying the assumption of the induction, we have

$$[\mathcal{X}^-_{(a,c),s}, \mathcal{E}^t_{(i,k),(j,l)}]$$
$$= \begin{cases} [\mathcal{X}^+_{(i,k),0}, \mathcal{E}^{t+s}_{(i+2,k),(j,l)}] & \text{if } (a,c) = (i+1,k) \text{ and } i+1 \neq m_k, \\ [\mathcal{X}^+_{(i,k),0}, -Q_k \mathcal{E}^{t+s}_{(1,k+1),(j,l)} + \mathcal{E}^{t+s+1}_{(1,k+1),(j,l)}] & \text{if } (a,c) = (i+1,k) \text{ and } i+1 = m_k \\ [\mathcal{X}^+_{(i,k),0}, -\mathcal{E}^{t+s}_{(i+1,k),(j-1,l)}] & \text{if } (a,c) = (j-1,l) \text{ and } j-1 \neq m_l, \\ [\mathcal{X}^+_{(i,k),0}, Q_l \mathcal{E}^{t+s}_{(i+1,k),(m_l,l)} - \mathcal{E}^{t+s+1}_{(i+1,k),(m_l,l)}] & \text{if } (a,c) = (j-1,l) \text{ and } j-1 = m_l, \\ [-\mathcal{I}_{(i,k),s} + \mathcal{I}_{(i+1,k),s}, \mathcal{E}^t_{(i+1,k),(j,l)}] & \text{if } (a,c) = (i,k) \text{ and } i \neq m_k, \\ [Q_k(\mathcal{I}_{(m_k,k),s} - \mathcal{I}_{(1,k+1),s}) - \mathcal{I}_{(m_k,k),s+1} + \mathcal{I}_{(1,k+1),s+1}, \mathcal{E}^t_{(1,k+1),(j,l)}] & \\ & \text{if } (a,c) = (i,k) \text{ and } i = m_k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, we have (2.5.3) by applying (2.5.1) and (2.5.2).

(ii) is proven in a similar way. (iii) is just the relations (L1) and (L2). $\square$

By Lemma 2.5, we see that $\mathfrak{g}$ is spanned by $\{\mathcal{E}^t_{(i,k)(j,l)} \,|\, (i,k),(j,l) \in \varGamma(\mathbf{m}), t \geq 0\}$ as a $\mathbb{Q}(\mathbf{Q})$-vector space. In fact, we see that it is a basis of $\mathfrak{g}$ as follows.

Proposition 2.6. $\{\mathcal{E}^t_{(i,k)(j,l)} \,|\, (i,k),(j,l) \in \varGamma(\mathbf{m}), t \geq 0\}$ *gives a basis of* $\mathfrak{g} = \mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$.

Proof. It is enough to show that $\{\mathcal{E}^t_{(i,k),(j,l)} \,|\, (i,k),(j,l) \in \varGamma(\mathbf{m}), t \geq 0\}$ are linearly independent.

For $\tau \in \mathbb{Q}(\mathbf{Q})$, let $V_\tau = \bigoplus_{(j,l)\in\varGamma(\mathbf{m})} \mathbb{Q}(\mathbf{Q}) v_{(j,l)}$ be the $\mathfrak{g}$-module given in 2.3. Then, we see that

$$\mathcal{E}^t_{(i,k)(j,l)} \cdot v_{(a,c)} = \delta_{(a,c)(j,l)} \psi_{(i,k)(j,l)} \tau^t v_{(i,k)},$$

where we put

$$\psi_{(i,k)(j,l)} = \begin{cases} \prod_{p=0}^{l-k-1} (-Q_l + \tau) & \text{if } l-k > 0, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, if $\sum_{(i,k),(j,l)\in\varGamma(\mathbf{m}),t\geq 0} r^t_{(i,k)(j,l)} \mathcal{E}^t_{(i,k),(j,l)} = 0$ ($r^t_{(i,k)(j,l)} \in \mathbb{Q}(\mathbf{Q})$), we have

$$\Big(\sum_{(i,k),(j,l)\in\varGamma(\mathbf{m}),t\geq 0} r^t_{(i,k)(j,l)} \mathcal{E}^t_{(i,k),(j,l)}\Big) \cdot v_{(a,c)} = \sum_{(i,k)\in\varGamma(\mathbf{m})} \psi_{(i,k)(j,l)} \Big(\sum_{t\geq 0} r^t_{(i,k)(a,c)} \tau^t\Big) v_{(i,k)}$$
$$= 0.$$

Thus, for any $(i,k),(j,l) \in \varGamma(\mathbf{m})$ and any $\tau \in \mathbb{Q}(\mathbf{Q})$, we have

$$\psi_{(i,k)(j,l)}\big(\sum_{t\geq 0} r^t_{(i,k)(j,l)}\tau^t\big) = 0.$$

This implies that $r^t_{(i,k)(j,l)} = 0$ for any $(i,k),(j,l) \in \varGamma(\mathbf{m})$ and any $t \geq 0$. $\square$

2.7. Let $\mathfrak{n}^+$, $\mathfrak{n}^-$ and $\mathfrak{n}^0$ be the Lie subalgebras of $\mathfrak{g}$ generated by

$$\{\mathcal{X}^+_{(i,k),t} \,|\, (i,k) \in \varGamma'(\mathbf{m}), t \geq 0\},\ \{\mathcal{X}^-_{(i,k),t} \,|\, (i,k) \in \varGamma'(\mathbf{m}), t \geq 0\} \text{ and}$$
$$\{\mathcal{I}_{(j,l),t} \,|\, (j,l) \in \varGamma(\mathbf{m}), t \geq 0\}$$

respectively. Then, we have the following triangular decomposition as a corollary of Proposition 2.6.

Corollary 2.8. *We have the triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{n}^0 \oplus \mathfrak{n}^+ \quad (\textit{as vector spaces}).$$

2.9. A current Lie algebra. Let $\mathbb{Q}[x]$ be the polynomial ring over $\mathbb{Q}$, and let $\mathfrak{gl}_m[x] = \mathbb{Q}[x] \otimes \mathfrak{gl}_m$ be the current Lie algebra associated with the general linear Lie algebra $\mathfrak{gl}_m$ over $\mathbb{Q}$. Namely, the Lie bracket on $\mathfrak{gl}_m[x]$ is defined by

$$[a \otimes g, b \otimes h] = ab \otimes [g,h] \quad (a,b \in \mathbb{Q}[x],\ g,h \in \mathfrak{gl}_m).$$

Let $E_{i,j} \in \mathfrak{gl}_m$ $(1 \leq i,j \leq m)$ be the elementary matrix having 1 at the (i,j)-entry and 0 elsewhere. Put $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$ and $K_j = E_{j,j}$. Then $\mathbb{Q}[x] \otimes \mathfrak{gl}_m$ is generated by

$$x^t \otimes e_i,\ x^t \otimes f_i,\ x^t \otimes K_j \quad (1 \leq i \leq m-1,\ 1 \leq j \leq m,\ t \geq 0).$$

2.10. In the case where $r = 1$ ($\mathbf{m} = m$), the Lie algebra $\mathfrak{g}(m)$ over $\mathbb{Q}$ is generated by $\mathcal{X}^\pm_{i,t}$ and $\mathcal{I}_{j,t}$ $(1 \leq i \leq m-1,\ 1 \leq j \leq m,\ t \geq 0)$ with the defining relations (L1)-(L6) (for $(i,1) \in \varGamma(m)$, we denote $(i,1)$ by i simply). In this case, the relation (L3) is just

$$[\mathcal{X}^+_{i,t}, \mathcal{X}^-_{j,s}] = \delta_{i,j}(\mathcal{I}_{i,t} - \mathcal{I}_{i+1,t}).$$

Then, we have the following lemma.

Lemma 2.11. *There exists the isomorphism of Lie algebras*

$$\Phi : \mathfrak{g}(m) \to \mathfrak{gl}_m[x] \quad (\mathcal{X}^+_{i,t} \mapsto x^t \otimes e_i,\ \mathcal{X}^-_{i,t} \mapsto x^t \otimes f_i,\ \mathcal{I}_{j,t} \mapsto x^t \otimes K_j).$$

In particular, the relations (L1)-(L6) (in the case where $r = 1$) give a defining relations of $\mathfrak{gl}_m[x]$ through the isomorphism Φ.

Proof. We can show the well-definedness of the homomorphism Φ by checking the defining relations of $\mathfrak{g}(m)$ directly.

For $i,j \in \{1,\dots,m\}$ and $t \geq 0$, we see that $\Phi(\mathcal{E}^t_{i,j}) = x^t \otimes E_{i,j}$. Clearly, $\{x^t \otimes E_{i,j} \mid 1 \leq i,j \leq m,\ t \geq 0\}$ gives a basis of $\mathfrak{gl}_m[x]$. Thus, Proposition 2.6 implies that Φ is isomorphic. $\square$

2.12. In the case where $r \geq 2$, we can regard $\mathfrak{g} = \mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ as a deformation of the current Lie algebra $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{Q}} \mathfrak{gl}_m[x]$ as follows.

For $t \geq 0$, put

$$\mathcal{Y}_t = \{\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{I}_{(j,l),t} \mid (i,k) \in \varGamma'(\mathbf{m}), (j,l) \in \varGamma(\mathbf{m})\}.$$

Let $\mathfrak{g}_t$ be the $\mathbb{Q}(\mathbf{Q})$-subspace of $\mathfrak{g}$ spanned by

$$\{[Y_{t_1}, [Y_{t_2}, \dots, [Y_{t_{p-1}}, Y_{t_p}] \dots] \mid Y_{t_b} \in \mathcal{Y}_{t_b}, \sum_{b=1}^{p} t_b \geq t,\ p \geq 1\}.$$

Then, we have the sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \dots .$$

By the defining relations (L1)-(L6), we see that

$$[\mathfrak{g}_s, \mathfrak{g}_t] \subset \mathfrak{g}_{s+t} \quad (s,t \geq 0). \tag{2.12.1}$$

For $t \geq 0$, let $\sigma_t : \mathfrak{g}_t \to \mathfrak{g}_t/\mathfrak{g}_{t+1}$ be the natural surjection. By (2.12.1), we can define the structure as a Lie algebra on $\mathbf{gr}\,\mathfrak{g} = \bigoplus_{t \geq 0} \mathfrak{g}_t/\mathfrak{g}_{t+1}$ by

$$[\sigma_s(g), \sigma_t(h)] = \sigma_{s+t}([g,h]) \quad (g \in \mathfrak{g}_s, h \in \mathfrak{g}_t).$$

Then we see that, $\mathbf{gr}\,\mathfrak{g}$ is generated by

$$\sigma_t(\mathcal{X}^{\pm}_{(i,k),t}),\ \sigma_t(\mathcal{I}_{(j,l),t}) \quad ((i,k) \in \varGamma'(\mathbf{m}),\ (j,l) \in \varGamma(\mathbf{m}),\ t \geq 0),$$

and $\mathbf{gr}\,\mathfrak{g}$ has a basis $\{\sigma_t(\mathcal{E}^t_{(i,k),(j,l)}) \mid (i,k),(j,l) \in \varGamma(\mathbf{m}),\ t \geq 0\}$.

Proposition 2.13. *There exists the isomorphism of Lie algebras*

$$\Psi : \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{Q}} \mathfrak{gl}_m[x] \to \mathbf{gr}\,\mathfrak{g} = \bigoplus_{t \geq 0} \mathfrak{g}_t/\mathfrak{g}_{t+1}$$

such that

$$\begin{aligned} x^t \otimes e_{(i,k)} &\mapsto \begin{cases} \sigma_t(\mathcal{X}^+_{(i,k),t}) & \text{if } i \neq m_k, \\ -Q_k^{-1}\sigma_t(\mathcal{X}^+_{(m_k,k),t}) & \text{if } i = m_k, \end{cases} \\ x^t \otimes f_{(i,k)} &\mapsto \sigma_t(\mathcal{X}^-_{(i,k),t}), \end{aligned}$$

$$x^t \otimes K_{(j,l)} \mapsto \sigma_t(\mathcal{I}_{(j,l),t}),$$

where we use the identification (1.3.1) *for the indices of generators of* $\mathfrak{gl}_m[x]$.

Proof. We can show the well-definedness of the homomorphism Ψ by checking the defining relations of $\mathfrak{gl}_m[x]$ directly (see Lemma 2.11). We also see that

$$\Psi(x^t \otimes E_{(i,k),(j,l)}) = \psi_{(i,k)(j,l)}\sigma_t(\mathcal{E}^t_{(i,k),(j,l)}),$$

where we put

$$\psi_{(i,k)(j,l)} = \begin{cases} \prod_{p=0}^{l-k-1}(-Q_{k+p}^{-1}) & \text{if } l-k>0, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, we see that Ψ is isomorphic. $\square$

As a corollary of the above proposition, we have the following isomorphism between $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{Q}} \mathfrak{g}(m)$ and $\mathbf{gr}\,\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$.

Corollary 2.14. *There exists the isomorphism of Lie algebras*

$$\widetilde{\Psi} : \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{Q}} \mathfrak{g}(m) \to \mathbf{gr}\,\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}) = \bigoplus_{t\geq 0} \mathfrak{g}_t/\mathfrak{g}_{t+1}$$

such that

$$\mathcal{X}^+_{(i,k),t} \mapsto \begin{cases} \sigma_t(\mathcal{X}^+_{(i,k),t}) & \textit{if } i \neq m_k, \\ -Q_k^{-1}\sigma_t(\mathcal{X}^+_{(m_k,k),t} & \textit{if } i = m_k, \end{cases} \quad \mathcal{X}^-_{(i,k),t} \mapsto \sigma_t(\mathcal{X}^-_{(i,k),t}),\ \mathcal{I}_{(j,l),t} \mapsto \sigma_t(\mathcal{I}_{(j,l),t}),$$

where we use the identification (1.3.1) *for the indices of generators of* $\mathfrak{g}(m)$.

2.15. We also have some relations between the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ and the general linear Lie algebra $\mathfrak{gl}_m$ as follows. Let $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ be a Levi subalgebra of $\mathfrak{gl}_m$ associated with $\mathbf{m} = (m_1, \dots, m_r)$. Then generates of $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ are given by $e_{(i,k)}$, $f_{(i,k)}$ $(1 \leq i \leq m_k - 1, 1 \leq k \leq r)$ and $K_{(j,l)}$ $((j,l) \in \Gamma(\mathbf{m}))$, where we use the identification (1.3.1) for indices.

Proposition 2.16.

(i) *There exists a surjective homomorphism of Lie algebras*

$$g : \mathfrak{g}_{\mathbf{Q}}(\mathbf{m}) \to \mathfrak{gl}_m \tag{2.16.1}$$

such that

$$g(\mathcal{X}^+_{(i,k),0}) = \begin{cases} e_{(i,k)} & \textit{if } i \neq m_k, \\ -Q_k e_{(m_k,k)} & \textit{if } i = m_k, \end{cases} \quad g(\mathcal{X}^-_{(i,k),0}) = f_{(i,k)},$$

$$g(\mathcal{I}_{(j,l),0}) = K_{(j,l)} \textit{ and } g(\mathcal{X}^{\pm}_{(i,k),t}) = g(\mathcal{I}_{(j,l),t}) = 0 \textit{ for } t \geq 1.$$

(ii) *There exists an injective homomorphism of Lie algebras*

$$\iota : \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \to \mathfrak{g}_{\mathbf{Q}}(\mathbf{m}) \tag{2.16.2}$$

such that $\iota(e_{(i,k)}) = \mathcal{X}^{+}_{(i,k),0}$, $\iota(f_{(i,k)}) = \mathcal{X}^{-}_{(i,k),0}$ *and* $\iota(K_{(j,l)}) = \mathcal{I}_{(j,l),0}$.

Proof. We can check the well-definedness of g and ι by direct calculations. Clearly g is surjective. Let $\iota' : \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \to \mathfrak{gl}_m$ be the natural embedding. Then, by investigating the image of generators, we see that $\iota' = g \circ \iota$. This implies that ι is injective. $\square$

Remark 2.17. The surjective homomorphism g in (2.16.1) can be regarded as a special case of evaluation homomorphisms. However, we can not define evaluation homomorphisms for $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ in general although we can consider $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$-modules corresponding to some evaluation modules. They will be studied in a subsequent paper.

§ 3. Representations of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$

Thanks to the triangular decomposition in Corollary 2.8, we can develop the weight theory to study some representations of $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ in the usual manner as follows.

3.1. Let $U(\mathfrak{g}) = U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$. Then, by Corollary 2.8 together with PBW theorem, we have the triangular decomposition

$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{n}^0) \otimes U(\mathfrak{n}^+). \tag{3.1.1}$$

Thanks to the triangular decomposition, we can develop the weight theory for $U(\mathfrak{g})$-modules as follows.

3.2. Highest weight modules. For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi_{(j,l),t} \mid (j,l) \in \varGamma(\mathbf{m}), t \geq 1)$ ($\varphi_{(j,l),t} \in \mathbb{Q}(\mathbf{Q})$), we say that a $U(\mathfrak{g})$-modules M is a highest weight modules of highest weight $(\lambda, \boldsymbol{\varphi})$ if there exists an element $v_0 \in M$ satisfying the following three conditions:

(i) M is generated by v_0 as a $U(\mathfrak{g})$-module,
(ii) $\mathcal{X}^{+}_{(i,k),t} \cdot v = 0$ for all $(i,k) \in \varGamma'(\mathbf{m})$ and $t \geq 0$,
(iii) $\mathcal{I}_{(j,l),0} \cdot v_0 = \langle \lambda, h_{(j,l)} \rangle v_0$ and $\mathcal{I}_{(j,l),t} \cdot v_0 = \varphi_{(j,l),t} v_0$ for $(j,l) \in \varGamma(\mathbf{m})$ and $t \geq 1$.

If an element $v_0 \in M$ satisfies the above conditions (ii) and (iii), we say that v_0 is a maximal vector of weight $(\lambda, \boldsymbol{\varphi})$. In this case, the submodule $U(\mathfrak{g}) \cdot v_0$ of M is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$. If a maximal vector $v_0 \in M$ satisfies the above condition (i), we say that v_0 is a highest weight vector.

For a highest weight $U(\mathfrak{g})$-module M of highest weight $(\lambda, \boldsymbol{\varphi})$ with a highest weight vector $v_0 \in M$, we have $M = U(\mathfrak{n}^-) \cdot v_0$ by the triangular decomposition

(3.1.1). Thus, the relation (L2) implies the weight space decomposition

$$M = \bigoplus_{\substack{\mu \in P \\ \mu \leq \lambda}} M_\mu \text{ such that } \dim_{\mathbb{Q}(\mathbf{Q})} M_\lambda = 1, \tag{3.2.1}$$

where $M_\mu = \{ v \in M \,|\, \mathcal{I}_{(j,l),0} \cdot v = \langle \mu, h_{(j,l)} \rangle v \text{ for } (j,l) \in \Gamma(\mathbf{m}) \}$.

3.3. Verma modules. Let $U(\mathfrak{n}^{\geq 0})$ be the subalgebra of $U(\mathfrak{g})$ generated by $U(\mathfrak{n}^0)$ and $U(\mathfrak{n}^+)$. Then, by Proposition 2.6 together with the proof of Lemma 2.5, we see that $U(\mathfrak{n}^+)$ (resp. $U(\mathfrak{n}^-)$) is isomorphic to the algebra generated by $\{\mathcal{X}^+_{(i,k),t} \,|\, (i,k) \in \Gamma'(\mathbf{m}), t \geq 0\}$ (resp. $\{\mathcal{X}^-_{(i,k),t} \,|\, (i,k) \in \Gamma'(\mathbf{m}), t \geq 0\}$) with the defining relations (L4)-(L6), $U(\mathfrak{n}^0)$ is isomorphic to the algebra generated by $\{\mathcal{I}_{(j,l),t} \,|\, (j,l) \in \Gamma(\mathbf{m}), t \geq 0\}$ with the defining relations (L1), and that $U(\mathfrak{n}^{\geq 0})$ is isomorphic to the algebra generated by $\{\mathcal{X}^+_{(i,k)t}, \mathcal{I}_{(j,l)t} \,|\, (i,k) \in \Gamma'(\mathbf{m}), (j,l) \in \Gamma(\mathbf{m}), t \geq 0\}$ with the defining relations (L1)-(L6) except (L3). Then we have the surjective homomorphism of algebras

$$U(\mathfrak{n}^{\geq 0}) \to U(\mathfrak{n}^0) \text{ such that } \mathcal{X}^+_{(i,k),t} \mapsto 0, \mathcal{I}_{(j,l),t} \mapsto \mathcal{I}_{(j,l),t}. \tag{3.3.1}$$

For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi_{(j,l),t})$, we define a (1-dimensional) simple $U(\mathfrak{n}^0)$-module $\Theta_{(\lambda,\boldsymbol{\varphi})} = \mathbb{Q}(\mathbf{Q}) v_0$ by

$$\mathcal{I}_{(j,l),0} \cdot v_0 = \langle \lambda, h_{(j,l)} \rangle v_0, \quad \mathcal{I}_{(j,l)t} \cdot v_0 = \varphi_{(j,l),t} v_0$$

for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$. Then we define the Verma module $M(\lambda, \boldsymbol{\varphi})$ as the induced module

$$M(\lambda, \boldsymbol{\varphi}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}^{\geq 0})} \Theta_{(\lambda,\boldsymbol{\varphi})},$$

where we regard $\Theta_{(\lambda,\boldsymbol{\varphi})}$ as a left $U(\mathfrak{n}^{\geq 0})$-module through the surjection (3.3.1).

By definitions, the Verma module $M(\lambda, \boldsymbol{\varphi})$ is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$ with a highest weight vector $1 \otimes v_0$. Then we see that any highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$ is a quotient of $M(\lambda, \boldsymbol{\varphi})$ by the universality of tensor products. We also see that $M(\lambda, \boldsymbol{\varphi})$ has the unique simple top $L(\lambda, \boldsymbol{\varphi}) = M(\lambda, \boldsymbol{\varphi}) / \operatorname{rad} M(\lambda, \boldsymbol{\varphi})$ from the weight space decomposition (3.2.1).

By using the homomorphism $\iota : U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U(\mathfrak{g})$ induced from (2.16.2), we have a necessary condition for $L(\lambda, \boldsymbol{\varphi})$ to be finite dimensional as follows.

Proposition 3.4. *For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi_{(j,l),t})$, if $L(\lambda, \boldsymbol{\varphi})$ is finite dimensional, then we have $\lambda \in P^+_{\mathbf{m}}$.*

Proof. Assume that $L(\lambda, \boldsymbol{\varphi})$ is finite dimensional. Let $v_0 \in L(\lambda, \boldsymbol{\varphi})$ be a highest weight vector. When we regard $L(\lambda, \boldsymbol{\varphi})$ as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module through the injection $\iota : U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U(\mathfrak{g})$, we see that $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-submodule of $L(\lambda, \boldsymbol{\varphi})$ generated by v_0 is a (finite dimensional) highest weight $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module of highest weight λ. Thus, the Lemma follows from the well-known facts for $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-modules. $\square$

3.5. Category $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$. Let $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{C}_{\mathbf{Q}}^{\geq 0}(\mathbf{m})$) be the full subcategory of $U(\mathfrak{g})$-mod consisting of $U(\mathfrak{g})$-modules satisfying the following conditions:

(i) If $M \in \mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}_{\mathbf{Q}}^{\geq 0}(\mathbf{m})$), then M is finite dimensional,

(ii) If $M \in \mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}_{\mathbf{Q}}^{\geq 0}(\mathbf{m})$), then M has the weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda \quad (\text{resp. } M = \bigoplus_{\lambda \in P_{\geq 0}} M_\lambda),$$

where $M_\lambda = \{v \in M \mid \mathcal{I}_{(j,l),0} \cdot v = \langle \lambda, h_{(j,l)} \rangle v \text{ for } (j,l) \in \Gamma(\mathbf{m})\}$,

(iii) If $M \in \mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}_{\mathbf{Q}}^{\geq 0}(\mathbf{m})$), then all eigenvalues of the action of $\mathcal{I}_{(j,l),t}$ ($(j,l) \in \Gamma(\mathbf{m}), t \geq 0$) on M belong to $\mathbb{Q}(\mathbf{Q})$.

By the usual argument, we have the following lemma.

Lemma 3.6. *Any simple object in $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ is a highest weight module.*

By using the surjection $g : U(\mathfrak{g}) \to U(\mathfrak{gl}_m)$ induced from (2.16.1), we have the following proposition.

Proposition 3.7. *Let $\mathcal{C}_{\mathfrak{gl}_m}$ be the category of finite dimensional $U(\mathfrak{gl}_m)$-modules which have the weight space decomposition. Then, we have the followings.*

(i) *$\mathcal{C}_{\mathfrak{gl}_m}$ is a full subcategory of $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ through the surjection $g : U(\mathfrak{g}) \to U(\mathfrak{gl}_m)$.*

(ii) *For $\lambda \in P^+$, the simple highest weight $U(\mathfrak{gl}_m)$-module $\Delta_{\mathfrak{gl}_m}(\lambda)$ of highest weight λ is the simple highest weight $U(\mathfrak{g})$-module of highest weight $(\lambda, \mathbf{0})$ through the surjection $g : U(\mathfrak{g}) \to U(\mathfrak{gl}_m)$, where $\mathbf{0}$ means $\varphi_{(j,l),t} = 0$ for all $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.*

§ 4. Algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$

In this section, we introduce an algebra $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ with parameters q and $\mathbf{Q} = (Q_1, \dots, Q_{r-1})$ associated with the Cartan data in the paragraph 1.3. Then we study some basic structures of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. In particular, we can regard $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a "q-analogue" of the universal enveloping algebra $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ introduced in the section §2.

4.1. Put $\mathbb{A} = \mathbb{Z}[\mathbf{Q}][q, q^{-1}] = \mathbb{Z}[q, q^{-1}, Q_1, \dots, Q_{r-1}]$, where $q, Q_1, \dots, Q_{r-1}$ are indeterminate elements over $\mathbb{Z}$, and let $\mathbb{K} = \mathbb{Q}(q, Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{A}$.

Definition 4.2. *We define the associative algebra $\mathcal{U} = \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ over $\mathbb{K}$ by the following generators and defining relations:*

Generators: $\mathcal{X}^{\pm}_{(i,k),t}$, $\mathcal{I}^{\pm}_{(j,l),t}$, $\mathcal{K}^{\pm}_{(j,l)}$ *($(i,k) \in \Gamma'(\mathbf{m})$, $(j,l) \in \Gamma(\mathbf{m})$, $t \geq 0$).*

Relations:

(R1)

$$\mathcal{K}^+_{(j,l)} \mathcal{K}^-_{(j,l)} = \mathcal{K}^-_{(j,l)} \mathcal{K}^+_{(j,l)} = 1, \quad (\mathcal{K}^{\pm}_{(j,l)})^2 = 1 \pm (q - q^{-1}) \mathcal{I}^{\mp}_{(j,l),0},$$

(R2)

$$[\mathcal{K}^+_{(i,k)}, \mathcal{K}^+_{(j,l)}] = [\mathcal{K}^+_{(i,k)}, \mathcal{I}^{\sigma}_{(j,l),t}] = [\mathcal{I}^{\sigma}_{(i,k),s}, \mathcal{I}^{\sigma'}_{(j,l),t}] = 0 \ (\sigma, \sigma' \in \{+,-\}),$$

(R3)

$$\mathcal{K}^+_{(j,l)} \mathcal{X}^{\pm}_{(i,k),t} \mathcal{K}^-_{(j,l)} = q^{\pm a_{(i,k)(j,l)}} \mathcal{X}^{\pm}_{(i,k),t},$$

(R4)

$$q^{\pm a_{(i,k)(j,l)}} \mathcal{I}^{\pm}_{(j,l),0} \mathcal{X}^+_{(i,k),t} - q^{\mp a_{(i,k)(j,l)}} \mathcal{X}^+_{(i,k),t} \mathcal{I}^{\pm}_{(j,l),0} = a_{(i,k)(j,l)} \mathcal{X}^+_{(i,k),t},$$
$$q^{\mp a_{(i,k)(j,l)}} \mathcal{I}^{\pm}_{(j,l),0} \mathcal{X}^-_{(i,k),t} - q^{\pm a_{(i,k)(j,l)}} \mathcal{X}^-_{(i,k),t} \mathcal{I}^{\pm}_{(j,l),0} = -a_{(i,k)(j,l)} \mathcal{X}^-_{(i,k),t},$$

(R5)

$$[\mathcal{I}^{\pm}_{(j,l),s+1}, \mathcal{X}^+_{(i,k),t}] = q^{\pm a_{(i,k)(j,l)}} \mathcal{I}^{\pm}_{(j,l),s} \mathcal{X}^+_{(i,k),t+1} - q^{\mp a_{(i,k)(j,l)}} \mathcal{X}^+_{(i,k),t+1} \mathcal{I}^{\pm}_{(j,l),s},$$
$$[\mathcal{I}^{\pm}_{(j,l),s+1}, \mathcal{X}^-_{(i,k),t}] = q^{\mp a_{(i,k)(j,l)}} \mathcal{I}^{\pm}_{(j,l),s} \mathcal{X}^-_{(i,k),t+1} - q^{\pm a_{(i,k)(j,l)}} \mathcal{X}^-_{(i,k),t+1} \mathcal{I}^{\pm}_{(j,l),s},$$

(R6)

$$[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(j,l),s}]$$
$$= \delta_{(i,k),(j,l)} \begin{cases} \widetilde{\mathcal{K}}^+_{(i,k)} \mathcal{J}_{(i,k),s+t} & \textit{if } i \neq m_k, \\ -Q_k \widetilde{\mathcal{K}}^+_{(m_k,k)} \mathcal{J}_{(m_k,k),s+t} + \widetilde{\mathcal{K}}^+_{(m_k,k)} \mathcal{J}_{(m_k,k),s+t+1} & \textit{if } i = m_k, \end{cases}$$

(R7)

$$[\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{X}^{\pm}_{(j,l),s}] = 0 \qquad \textit{if } (j,l) \neq (i,k), (i \pm 1, k),$$
$$\mathcal{X}^{\pm}_{(i,k),t+1} \mathcal{X}^{\pm}_{(i,k),s} - q^{\pm 2} \mathcal{X}^{\pm}_{(i,k),s} \mathcal{X}^{\pm}_{(i,k),t+1} = q^{\pm 2} \mathcal{X}^{\pm}_{(i,k),t} \mathcal{X}^{\pm}_{(i,k),s+1} - \mathcal{X}^{\pm}_{(i,k),s+1} \mathcal{X}^{\pm}_{(i,k),t},$$
$$\mathcal{X}^+_{(i,k),t+1} \mathcal{X}^+_{(i+1,k),s} - q^{-1} \mathcal{X}^+_{(i+1,k),s} \mathcal{X}^+_{(i,k),t+1} = \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i+1,k),s+1} - q \mathcal{X}^+_{(i+1,k),s+1} \mathcal{X}^+_{(i,k),t},$$
$$\mathcal{X}^-_{(i+1,k),s} \mathcal{X}^-_{(i,k),t+1} - q^{-1} \mathcal{X}^-_{(i,k),t+1} \mathcal{X}^-_{(i+1,k),s} = \mathcal{X}^-_{(i+1,k),s+1} \mathcal{X}^-_{(i,k),t} - q \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i+1,k),s+1},$$

(R8)

$$\mathcal{X}^+_{(i\pm1,k),u} \big(\mathcal{X}^+_{(i,k),s} \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i,k),s} \big) + \big(\mathcal{X}^+_{(i,k),s} \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i,k),s} \big) \mathcal{X}^+_{(i\pm1,k),u}$$
$$= (q + q^{-1}) \big(\mathcal{X}^+_{(i,k),s} \mathcal{X}^+_{(i\pm1,k),u} \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i\pm1,k),u} \mathcal{X}^+_{(i,k),s} \big),$$

$$\mathcal{X}^-_{(i\pm1,k),u} \big(\mathcal{X}^-_{(i,k),s} \mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i,k),s} \big) + \big(\mathcal{X}^-_{(i,k),s} \mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i,k),s} \big) \mathcal{X}^-_{(i\pm1,k),u}$$
$$= (q + q^{-1}) \big(\mathcal{X}^-_{(i,k),s} \mathcal{X}^-_{(i\pm1,k),u} \mathcal{X}^-_{(i,k),t} + \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i\pm1,k),u} \mathcal{X}^-_{(i,k),s} \big),$$

where we put $\widetilde{\mathcal{K}}^+_{(i,k)} = \mathcal{K}^+_{(i,k)} \mathcal{K}^-_{(i+1,k)}$, $\widetilde{\mathcal{K}}^-_{(i,k)} = \mathcal{K}^-_{(i,k)} \mathcal{K}^+_{(i+1,k)}$ *and*

$$\mathcal{J}_{(i,k),t} = \begin{cases} \mathcal{I}^+_{(i,k),0} - \mathcal{I}^-_{(i+1,k),0} + (q - q^{-1}) \mathcal{I}^+_{(i,k),0} \mathcal{I}^-_{(i+1,k),0} & \textit{if } t = 0, \\ q^{-t} \mathcal{I}^+_{(i,k),t} - q^t \mathcal{I}^-_{(i+1,k),t} - (q - q^{-1}) \displaystyle\sum_{b=1}^{t-1} q^{-t+2b} \mathcal{I}^+_{(i,k),t-b} \mathcal{I}^-_{(i+1,k),b} & \textit{if } t > 0. \end{cases}$$

Remark 4.3. The relations (R4) follows from the relations (R1) and (R3) in $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. Thus, we do not need the relations (R4) as a defining relations of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$. However, (R4) does not follows from (R1) and (R3) in the integral forms $\mathcal{U}^{\star}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ and $\mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ defined below. Then, we require the relations (R4) in a defining relations of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$.

4.4. By the relation (R1), for $(i,k) \in \varGamma'(\mathbf{m})$, we have

$$\widetilde{\mathcal{K}}^+_{(i,k)} \mathcal{J}_{(i,k),0} = \frac{\widetilde{\mathcal{K}}^+_{(i,k)} - \widetilde{\mathcal{K}}^-_{(i,k)}}{q-q^{-1}}. \tag{4.4.1}$$

Thus, in the case where $s=t=0$, we can replace the relation (R6) by

(4.4.2)

$$[\mathcal{X}^+_{(i,k),0}, \mathcal{X}^-_{(j,l),0}] = \delta_{(i,k),(j,l)} \begin{cases} \dfrac{\widetilde{\mathcal{K}}^+_{(i,k)} - \widetilde{\mathcal{K}}^-_{(i,k)}}{q-q^{-1}} & \text{if } i \neq m_k, \\ -Q_k \dfrac{\widetilde{\mathcal{K}}^+_{(m_k,k)} - \widetilde{\mathcal{K}}^-_{(m_k,k)}}{q-q^{-1}} + \widetilde{\mathcal{K}}^+_{(m_k,k)} \mathcal{J}_{(m_k,k),1} & \text{if } i = m_k. \end{cases}$$

By (R8), if $s=t$, we have

(4.4.3)

$$\begin{aligned} &\mathcal{X}^+_{(i\pm1,k),u} (\mathcal{X}^+_{(i,k),t})^2 - (q+q^{-1}) \mathcal{X}^+_{(i,k),t} \mathcal{X}^+_{(i\pm1,k),u} \mathcal{X}^+_{(i,k),t} + (\mathcal{X}^+_{(i,k),t})^2 \mathcal{X}^+_{(i\pm1,k),u} = 0, \\ &\mathcal{X}^-_{(i\pm1,k),u} (\mathcal{X}^-_{(i,k),t})^2 - (q+q^{-1}) \mathcal{X}^-_{(i,k),t} \mathcal{X}^-_{(i\pm1,k),u} \mathcal{X}^-_{(i,k),t} + (\mathcal{X}^-_{(i,k),t})^2 \mathcal{X}^-_{(i\pm1,k),u} = 0. \end{aligned}$$

By (R4) and (R5), we have

$$[\mathcal{I}^+_{(j,l),1}, \mathcal{X}^\pm_{(i,k),t}] = [\mathcal{I}^-_{(j,l),1}, \mathcal{X}^\pm_{(i,k),t}] = \pm a_{(i,k)(j,l)} \mathcal{X}^\pm_{(i,k),t+1}. \tag{4.4.4}$$

By the induction on s using the relation (R6), for $s \geq 1$, we can show that

(4.4.5)

$$\begin{aligned} &[\mathcal{I}^\pm_{(j,l),s}, \mathcal{X}^+_{(i,k),t}] \\ &= a_{(i,k)(j,l)} q^{\pm a_{(i,k)(j,l)}(s-1)} \mathcal{X}^+_{(i,k),t+s} \pm a_{(i,k)(j,l)} (q-q^{-1}) \sum_{p=1}^{s-1} q^{\pm a_{(i,k)(J,l)}(p-1)} \mathcal{X}^+_{(i,k),t+p} \mathcal{I}^\pm_{(j,l),s-p} \\ &= a_{(i,k)(j,l)} q^{\mp a_{(i,k)(j,l)}(s-1)} \mathcal{X}^+_{(i,k),t+s} \pm a_{(i,k)(j,l)} (q-q^{-1}) \sum_{p=1}^{s-1} q^{\mp a_{(i,k)(j,l)}(p-1)} \mathcal{I}^\pm_{(j,l),s-p} \mathcal{X}^+_{(i,k),t+p}, \end{aligned}$$

and

(4.4.6)

$$\begin{aligned} &[\mathcal{I}^\pm_{(j,l),s}, \mathcal{X}^-_{(i,k),t}] \\ &= -a_{(i,k)(j,l)} q^{\mp a_{(i,k)(j,l)}(s-1)} \mathcal{X}^-_{(i,k),t+s} \mp a_{(i,k)(j,l)} (q-q^{-1}) \sum_{p=1}^{s-1} q^{\mp a_{(i,k)(j,l)}(p-1)} \mathcal{X}^-_{(i,k),t+p} \mathcal{I}^\pm_{(j,l),s-p} \\ &= -a_{(i,k)(j,l)} q^{\pm a_{(i,k)(j,l)}(s-1)} \mathcal{X}^-_{(i,k),t+s} \mp a_{(i,k)(j,l)} (q-q^{-1}) \sum_{p=1}^{s-1} q^{\pm a_{(i,k)(j,l)}(p-1)} \mathcal{I}^\pm_{(j,l),s-p} \mathcal{X}^-_{(i,k),t+p}. \end{aligned}$$

4.5. Let $\mathcal{U}^+ = \mathcal{U}^+_{q,\mathbf{Q}}(\mathbf{m})$, $\mathcal{U}^- = \mathcal{U}^-_{q,\mathbf{Q}}(\mathbf{m})$ and $\mathcal{U}^0 = \mathcal{U}^0_{q,\mathbf{Q}}(\mathbf{m})$ be the subalgebra of $\mathcal{U}$ generated by

$$\begin{aligned}&\{\mathcal{X}^+_{(i,k),t} \,|\, (i,k) \in \Gamma'(\mathbf{m}),\ t \geq 0\},\\ &\{\mathcal{X}^-_{(i,k),t} \,|\, (i,k) \in \Gamma'(\mathbf{m}),\ t \geq 0\} \text{ and}\\ &\{\mathcal{I}^\pm_{(j,l),t},\ \mathcal{K}^\pm_{(j,l)} \,|\, (j,l) \in \Gamma(\mathbf{m}),\ t \geq 0\}\end{aligned}$$

respectively. Then, we have the following triangular decomposition of $\mathcal{U}$ from the relations (R1)-(R8), (4.4.5) and (4.4.6).

Proposition 4.6. *We have*

$$\mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+. \tag{4.6.1}$$

Remark 4.7. We conjecture that the multiplication map $\mathcal{U}^- \otimes_{\mathbb{K}} \mathcal{U}^0 \otimes_{\mathbb{K}} \mathcal{U}^+ \to \mathcal{U}$ ($x \otimes y \otimes z \mapsto xyz$) gives an isomorphism as vector spaces. More precisely, we expect the existence of a PBW type basis of $\mathcal{U}$ (cf. Proposition 2.6 and (4.11.2) with Remark 4.12).

4.8. We have some relations between the algebra $\mathcal{U}$ and a quantum group associated with the general linear Lie algebra as follows.

Let $U_q(\mathfrak{gl}_m)$ be the quantum group associated with the general linear Lie algebra $\mathfrak{gl}_m$ over $\mathbb{K}$. Namely, $U_q(\mathfrak{gl}_m)$ is an associative algebra over $\mathbb{K}$ generated by e_i, f_i ($1 \leq i \leq m-1$) and $K^\pm_j$ ($1 \leq j \leq m$) with the following defining relations:

$$K^+_i K^+_j = K^+_j K^+_i, \quad K^+_i K^-_i = K^-_i K^+_i = 1, \tag{Q1}$$

$$K^+_j e_i K^-_j = q^{a_{ij}} e_i, \quad K^+_j f_i K^-_j = q^{-a_{ij}} f_i, \text{ where } a_{ij} = \langle \alpha_i, h_j \rangle, \tag{Q2}$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{K^+_i K^-_{i+1} - K^-_i K^+_{i+1}}{q - q^{-1}}, \tag{Q3}$$

$$e_{i\pm1} e_i^2 - (q+q^{-1}) e_i e_{i\pm1} e_i + e_i^2 e_{i\pm1} = 0, \quad e_i e_j = e_j e_i \,(|i-j| \geq 2), \tag{Q4}$$

$$f_{i\pm1} f_i^2 - (q+q^{-1}) f_i f_{i\pm1} f_i + f_i^2 f_{i\pm1} = 0, \quad f_i f_j = f_j f_i \,(|i-j| \geq 2). \tag{Q5}$$

Let $U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \cong U_q(\mathfrak{gl}_{m_1}) \otimes \cdots \otimes U_q(\mathfrak{gl}_{m_r})$ be the Levi subalgebra of $U_q(\mathfrak{gl}_m)$ associated with the Levi subalgebra $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ of $\mathfrak{gl}_m$. Then generators of $U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$ are given by $e_{(i,k)}, f_{(i,k)}$ ($1 \leq i \leq m_k - 1, 1 \leq k \leq r$) and $K^\pm_{(j,l)}$ ($(j,l) \in \Gamma(\mathbf{m})$), where we use the identification (1.3.1) for indices.

Proposition 4.9.

(i) *There exits a surjective homomorphism of algebras*

$$g : \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \to U_q(\mathfrak{gl}_m) \tag{4.9.1}$$

such that

$$g(\mathcal{X}^+_{(i,k),0}) = \begin{cases} e_{(i,k)} & \text{if } i \neq m_k, \\ -Q_k e_{(m_k,k)} & \text{if } i = m_k, \end{cases} \quad g(\mathcal{X}^-_{(i,k),0}) = f_{(i,k)},$$
$$g(\mathcal{K}^\pm_{(j,l)}) = K^\pm_{(j,l)} \text{ and } g(\mathcal{X}^\pm_{(i,k),t}) = g(\mathcal{I}^\pm_{(j,l),t}) = 0 \text{ for } t \geq 1.$$

(ii) *There exists an injective homomorphism of algebras*

$$\iota : U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \tag{4.9.2}$$

such that $\iota(e_{(i,k)}) = \mathcal{X}^+_{(i,k),0}$*,* $\iota(f_{(i,k)}) = \mathcal{X}^-_{(i,k),0}$ *and* $\iota(K^\pm_{(j,l)}) = \mathcal{K}^\pm_{(j,l)}$*.*

Proof. We can check the well-definedness of g and ι by direct calculations. Clearly g is surjective. Let $\iota' : U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U_q(\mathfrak{gl}_m)$ be the natural embedding. Then, by investigating the image of generators, we see that $\iota' = g \circ \iota$. This implies that ι is injective. □

Remark 4.10. The surjective homomorphism g in (4.9.1) can be regarded as a special case of evaluation homomorphisms. However, we can not define evaluation homomorphisms for $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ in general although we can consider $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$-modules corresponding to some evaluation modules. They will be studied in a subsequent paper.

4.11. Let $\mathcal{U}^\star_{\mathbb{A}} = \mathcal{U}^\star_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-subalgebra of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ generated by

$$\{\mathcal{X}^\pm_{(i,k),t}, \mathcal{I}^\pm_{(j,l),t}, \mathcal{K}^\pm_{(j,l)} \mid (i,k) \in \Gamma'(\mathbf{m}), (j,l) \in \Gamma(\mathbf{m}), t \geq 0\}.$$

Then, $\mathcal{U}^\star_{\mathbb{A}}$ is an associative algebra over $\mathbb{A}$ generated by the same generators with the defining relations (R1)-(R8). We regard $\mathbb{Q}(\mathbf{Q})$ as an $\mathbb{A}$-module through the ring homomorphism $\mathbb{A} \to \mathbb{Q}(\mathbf{Q})$ $(q \mapsto 1)$, and we consider the specialization $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^\star_{\mathbb{A}}$ using this ring homomorphism. Let $\mathfrak{J}$ be the ideal of $\mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^\star_{\mathbb{A}}$ generated by

$$\{\mathcal{K}^+_{(j,l)} - 1, \mathcal{I}^+_{(j,l),t} - \mathcal{I}^-_{(j,l),t} \mid (i,l) \in \Gamma(\mathbf{m}), t \geq 0\}. \tag{4.11.1}$$

Let $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})$ defined in Definition 2.2. Then we can check that there exists a surjective homomorphism of algebras

$$U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m})) \to \mathbb{Q}(\mathbf{Q}) \otimes_{\mathbb{A}} \mathcal{U}^\star_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})/\mathfrak{J} \tag{4.11.2}$$

such that $\mathcal{X}^\pm_{(i,k),t} \mapsto \mathcal{X}^\pm_{(i,k),t}$ and $\mathcal{I}_{(j,l),t} \mapsto \mathcal{I}^+_{(j,l),t} (= \mathcal{I}^-_{(j,l),t})$.

Remark 4.12. We conjecture that the homomorphism (4.11.2) is isomorphic. Then we may regard $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ as a q-analogue of $U(\mathfrak{g}_{\mathbf{Q}}(\mathbf{m}))$.

We also remark that we have $(\mathcal{K}^+_{(j,l)})^2 = 1$ in $\mathcal{U}^\star_{\mathbb{A}}$ by the relation (R1). On the other hand, there exists an algebra automorphism of $\mathcal{U}$ such that $\mathcal{K}^\pm_{(j,l)} \mapsto -\mathcal{K}^\pm_{(j,l)}$

and the other generators send to the same generators. Thus, the choice of signs for $\mathcal{K}^+_{(j,l)}$ in (4.11.1) will not cause any troubles.

4.13. The final of this section, we define the $\mathbb{A}$-form of $\mathcal{U}$ taking the divided powers.

For $(i,k) \in \Gamma'(\mathbf{m})$ and $t, d \in \mathbb{Z}_{\geq 0}$, put

$$\mathcal{X}^{\pm(d)}_{(i,k),t} = \frac{(\mathcal{X}^{\pm}_{(i,k),t})^d}{[d]!} \in \mathcal{U}.$$

For $(j,l) \in \Gamma(\mathbf{m})$ and $d \in \mathbb{Z}_{\geq 0}$, put

$$\begin{bmatrix} \mathcal{K}_{(j,l)}; 0 \\ d \end{bmatrix} = \prod_{b=1}^{d} \frac{\mathcal{K}^+_{(j,l)} q^{-b+1} - \mathcal{K}^-_{(j,l)} q^{b-1}}{q^b - q^{-b}} \in \mathcal{U}.$$

Let $\mathcal{U}_{\mathbb{A}} = \mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m})$ be the $\mathbb{A}$-subalgebra of $\mathcal{U}$ generated by all $\mathcal{X}^{\pm(d)}_{(i,k),t}$, $\mathcal{I}^{\pm}_{(j,l),t}$, $\mathcal{K}^{\pm}_{(j,l)}$ and $\left[\begin{smallmatrix} \mathcal{K}_{(j,l)};0 \\ d \end{smallmatrix}\right]$.

§ 5. Representations of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$

Thanks to the triangular decomposition (4.6.1) of $\mathcal{U} = \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$, we can develop the weight theory to study $\mathcal{U}$-modules in the usual manner as follows.

5.1. Highest weight modules. For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi^{\pm}_{(j,l),t} \,|\, (j,l) \in \Gamma(\mathbf{m}), t \geq 1)$ ($\varphi^{\pm}_{(j,l),t} \in \mathbb{K}$), we say that a $\mathcal{U}$-module M is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$ if there exists an element $v_0 \in M$ satisfying the following three conditions:

(i) M is generated by v_0 as a $\mathcal{U}$-module,
(ii) $\mathcal{X}^+_{(i,k),t} \cdot v_0 = 0$ for all $(i,k) \in \Gamma'(\mathbf{m})$ and $t \geq 0$,
(iii) $\mathcal{K}^+_{(j,l)} \cdot v_0 = q^{\langle \lambda, h_{(j,l)} \rangle} v_0$ and $\mathcal{I}^{\pm}_{(j,l),t} \cdot v_0 = \varphi^{\pm}_{(j,l),t} v_0$ for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.

If an element $v_0 \in M$ satisfies the above conditions (ii) and (iii), we say that v_0 is a maximal vector of weight $(\lambda, \boldsymbol{\varphi})$. In this case, the submodule $\mathcal{U} \cdot v_0$ of M is a highest weight module of highest weight $(\lambda, \boldsymbol{\varphi})$. If a maximal vector $v_0 \in M$ satisfies also the above condition (i), we say that v_0 is a highest weight vector.

If $v_0 \in M$ is a maximal vector of weight $(\lambda, \boldsymbol{\varphi})$, for $(j,l) \in \Gamma(\mathbf{m})$, we have

$$\mathcal{I}^{\pm}_{(j,l),0} \cdot v = q^{\mp \lambda_{(j,l)}} [\lambda_{(j,l)}] v, \text{ where } \lambda_{(j,l)} = \langle \lambda, h_{(j,l)} \rangle$$

by the relation (R1).

For a highest weight $\mathcal{U}$-module M of highest weight $(\lambda, \boldsymbol{\varphi})$ with a highest weight vector $v_0 \in M$, we have $M = \mathcal{U}^- \cdot v_0$ by the triangular decomposition (4.6.1). Thus, the relation (R3) implies the weight space decomposition

$$M = \bigoplus_{\substack{\mu \in P \\ \mu \leq \lambda}} M_\mu \text{ such that } \dim_{\mathbb{K}} M_\lambda = 1, \tag{5.1.1}$$

where $M_\mu = \{v \in M \mid \mathcal{K}^+_{(j,l)} \cdot v = q^{\langle \mu, h_{(j,l)}\rangle} v \text{ for } (j,l) \in \Gamma(\mathbf{m})\}$.

5.2. Verma modules. Let $\widetilde{\mathcal{U}}^0$ be the associative algebra over $\mathbb{K}$ generated by $\mathcal{I}^\pm_{(j,l),t}$ and $\mathcal{K}^\pm_{(j,l)}$ for all $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$ with the defining relations (R1) and (R2). We also define the associative algebra $\widetilde{\mathcal{U}}^{\geq 0}$ generated by $\mathcal{X}^+_{(i,k),t}$, $\mathcal{I}^\pm_{(j,l),t}$ and $\mathcal{K}^\pm_{(j,l)}$ for all $(i,k) \in \Gamma'(\mathbf{m})$, $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$ with the defining relations (R1)-(R8) except (R6). Then we have the homomorphism of algebras

$$\widetilde{\mathcal{U}}^{\geq 0} \to \mathcal{U} \text{ such that } \mathcal{X}^+_{(i,k),t} \mapsto \mathcal{X}^+_{(i,k),t}, \mathcal{I}^\pm_{(j,l),t} \mapsto \mathcal{I}^\pm_{(j,l),t}, \tag{5.2.1}$$

and the surjective homomorphism of algebras

$$\widetilde{\mathcal{U}}^{\geq 0} \to \widetilde{\mathcal{U}}^0 \text{ such that } \mathcal{X}^+_{(i,k),t} \mapsto 0, \mathcal{I}^\pm_{(j,l)} \mapsto \mathcal{I}^\pm_{(j,l),t}, \mathcal{K}^\pm_{(j,l)} \mapsto \mathcal{K}^\pm_{(j,l)}. \tag{5.2.2}$$

For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi^\pm_{(j,l),t})$, we define a (1-dimensional) simple $\widetilde{\mathcal{U}}^0$-module $\Theta_{(\lambda,\boldsymbol{\varphi})} = \mathbb{K}v_0$ by

$$\mathcal{K}^+_{(j,l)} \cdot v_0 = q^{\langle \lambda, h_{(j,l)}\rangle} v_0, \quad \mathcal{I}^\pm_{(j,l),t} \cdot v_0 = \varphi^\pm_{(j,l),t} v_0$$

for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$. Then we define the Verma module $M(\lambda,\boldsymbol{\varphi})$ as the induced module

$$M(\lambda,\boldsymbol{\varphi}) = \mathcal{U} \otimes_{\widetilde{\mathcal{U}}^{\geq 0}} \Theta_{(\lambda,\boldsymbol{\varphi})},$$

where we regard $\Theta_{(\lambda,\boldsymbol{\varphi})}$ (resp. $\mathcal{U}$) as a left (resp. right) $\widetilde{\mathcal{U}}^{\geq 0}$-module through the homomorphism (5.2.2) (resp. (5.2.1)).

By definitions, the Verma module $M(\lambda,\boldsymbol{\varphi})$ is a highest weight module of highest weight $(\lambda,\boldsymbol{\varphi})$ with a highest weight vector $1 \otimes v_0$. Then we see that any highest weight module of highest weight $(\lambda,\boldsymbol{\varphi})$ is a quotient of $M(\lambda,\boldsymbol{\varphi})$ by the universality of tensor products. We also see that $M(\lambda,\boldsymbol{\varphi})$ has the unique simple top $L(\lambda,\boldsymbol{\varphi}) = M(\lambda,\boldsymbol{\varphi})/\operatorname{rad} M(\lambda,\boldsymbol{\varphi})$ from the weight space decomposition (5.1.1).

By using the homomorphism $\iota : U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to \mathcal{U}$ in (4.9.2), we have the following necessary condition for $L(\lambda,\boldsymbol{\varphi})$ to be finite dimensional in a similar way as in the proof of Proposition 3.4.

Proposition 5.3. *For $\lambda \in P$ and a multiset $\boldsymbol{\varphi} = (\varphi^\pm_{(j,l),t})$, if $L(\lambda,\boldsymbol{\varphi})$ is finite dimensional, then we have $\lambda \in P^+_{\mathbf{m}}$.*

5.4. Category $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$. Let $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$) be the full subcategory of $\mathcal{U}$-mod consisting of $\mathcal{U}$-modules satisfying the following conditions:

(i) If $M \in \mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$), then M is finite dimensional,

(ii) If $M \in \mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$), then M has the weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda \quad (\text{resp. } M = \bigoplus_{\lambda \in P_{\geq 0}} M_\lambda),$$

where $M_\lambda = \{v \in M \mid \mathcal{K}^+_{(j,l)} \cdot m = q^{\langle \lambda, h_{(j,l)} \rangle} v \text{ for } (j,l) \in \Gamma(\mathbf{m})\}$,

(iii) If $M \in \mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$), then all eigenvalues of the action of $\mathcal{I}^{\pm}_{(j,l),t}$ $((j,l) \in \Gamma(\mathbf{m}), t \geq 0)$ on M belong to $\mathbb{K}$.

By the usual argument, we have the following lemma.

Lemma 5.5. *Any simple object in $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ is a highest weight module.*

By using the surjection $g : \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \to U_q(\mathfrak{gl}_m)$ in (4.9.1), we have the following proposition.

Proposition 5.6. *Let $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ be the category of finite dimensional $U_q(\mathfrak{gl}_m)$-modules which have the weight space decomposition. Then we have the followings.*

(i) *$\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ is a full subcategory of $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ through the surjection* (4.9.1).

(ii) *For $\lambda \in P^+$, the simple highest weight $U_q(\mathfrak{gl}_m)$-module $\Delta_{U_q(\mathfrak{gl}_m)}(\lambda)$ of highest weight λ is the simple highest weight $\mathcal{U}$-module of highest weight $(\lambda, \mathbf{0})$ through the surjection* (4.9.1), *where $\mathbf{0}$ means $\varphi^{\pm}_{(j,l),t} = 0$ for all $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 1$.*

§ 6. Review of cyclotomic q-Schur algebras

In this section, we recall the definition and some fundamental properties of the cyclotomic q-Schur algebra $\mathscr{S}_{n,r}(\mathbf{m})$ introduced in [DJM]. See [DJM] and [M1] for details.

6.1. Let R be a commutative ring, and we take parameters $q, Q_0, Q_1, \dots, Q_{r-1} \in R$ such that q is invertible in R. The Ariki-Koike algebra $\mathscr{H}_{n,r}$ associated with the complex reflection group $\mathfrak{S}_n \ltimes (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over R generated by $T_0, T_1, \dots, T_{n-1}$ with the following defining relations:

$$\begin{aligned}
&(T_0 - Q_0)(T_0 - Q_1)\dots(T_0 - Q_{r-1}) = 0, \quad (T_i - q)(T_i + q^{-1}) = 0 \quad (1 \leq i \leq n-1),\\
&T_0T_1T_0T_1 = T_1T_0T_1T_0, \quad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad (1 \leq i \leq n-2),\\
&T_iT_j = T_jT_i \quad (|i-j| \geq 2).
\end{aligned}$$

The subalgebra of $\mathscr{H}_{n,r}$ generated by $T_1, \dots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra $\mathscr{H}_n$ associated with the symmetric group $\mathfrak{S}_n$ of degree n. For $w \in \mathfrak{S}_n$, we denote by $\ell(w)$ the length of w, and denote by T_w the standard basis of $\mathscr{H}_n$ corresponding to w.

6.2. Put $L_1 = T_0$ and $L_i = T_{i-1}L_{i-1}T_{i-1}$ for $i = 2, \dots, n$. These elements $L_1, \dots, L_n$ are called Jucys-Murphy elements of $\mathscr{H}_{n,r}$ (see [M2] for properties of Jucys-Murphy elements). The following lemma is well-known, and one can easily check them from defining relations of $\mathscr{H}_{n,r}$.

Lemma 6.3. *We have the following.*

(i) L_i *and* L_j *commute with each other for any* $1 \le i, j \le n$.
(ii) T_i *and* L_j *commute with each other if* $j \ne i, i+1$.
(iii) T_i *commutes with both* L_iL_{i+1} *and* $L_i + L_{i+1}$ *for any* $1 \le i \le n-1$.
(iv) $L_{i+1}^t T_i = (q - q^{-1}) \sum_{s=0}^{t-1} L_{i+1}^{t-s} L_i^s + T_i L_i^t$ *for any* $1 \le i \le n-1$ *and* $t \ge 1$.
(v) $L_i^t T_i = -(q - q^{-1}) \sum_{s=1}^{t} L_i^{t-s} L_{i+1}^s + T_i L_{i+1}^t$ *for any* $1 \le i \le n-1$ *and* $t \ge 1$.

6.4. Let $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{>0}^r$ be an r-tuple of positive integers. Put

$$\varLambda_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(r)}) \,\middle|\, \begin{array}{c} \mu^{(k)} = (\mu_1^{(k)}, \dots, \mu_{m_k}^{(k)}) \in \mathbb{Z}_{\ge 0}^{m_k} \\ \sum_{k=1}^r \sum_{i=1}^{m_k} \mu_i^{(k)} = n \end{array} \right\}.$$

We also put

$$\varLambda_{n,r}^+(\mathbf{m}) = \{ \mu \in \varLambda_{n,r}(\mathbf{m}) \mid \mu_1^{(k)} \ge \mu_2^{(k)} \ge \dots \ge \mu_{m_k}^{(k)} \ge 0 \text{ for each } k = 1, \dots, r \}.$$

We regard $\varLambda_{n,r}(\mathbf{m})$ as a subset of weight lattice $P = \bigoplus_{(i,k) \in \varGamma(\mathbf{m})} \mathbb{Z}\varepsilon_{(i,k)}$ by the injection $\varLambda_{n,r}(\mathbf{m}) \to P$ such that $\mu \mapsto \sum_{(i,k) \in \varGamma(\mathbf{m})} \mu_i^{(k)} \varepsilon_{(i,k)}$. Then we see that $\varLambda_{n,r}^+(\mathbf{m}) = \varLambda_{n,r}(\mathbf{m}) \cap P_{\mathbf{m}}^+$.

For $\mu \in \varLambda_{n,r}(\mathbf{m})$, put

$$m_\mu = \Big(\sum_{w \in \mathfrak{S}_\mu} q^{\ell(w)} T_w \Big) \Big(\prod_{k=1}^{r-1} \prod_{i=1}^{a_k} (L_i - Q_k) \Big), \tag{6.4.1}$$

where $\mathfrak{S}_\mu$ is the Young subgroup of $\mathfrak{S}_n$ with respect to μ, and $a_k = \sum_{j=1}^k |\mu^{(j)}|$. The following fact is well known:

$$m_\mu T_w = q^{\ell(w)} m_\mu \text{ if } w \in \mathfrak{S}_\mu. \tag{6.4.2}$$

The cyclotomic q-Schur algebra $\mathscr{S}_{n,r}(\mathbf{m})$ associated with $\mathscr{H}_{n,r}$ is defined by

$$\mathscr{S}_{n,r}(\mathbf{m}) = \operatorname{End}_{\mathscr{H}_{n,r}} \Big(\bigoplus_{\mu \in \varLambda_{n,r}(\mathbf{m})} m_\mu \mathscr{H}_{n,r} \Big). \tag{6.4.3}$$

For convenience in the later arguments, put $m_\mu = 0$ for $\mu \in P \setminus \varLambda_{n,r}(\mathbf{m})$.

6.5. Put $\widetilde{\varLambda}_{n,r}^+(\mathbf{m}) = \varLambda_{n,r}^+((n, \dots, n, m_r))$. It is clear that $\widetilde{\varLambda}_{n,r}^+(\mathbf{m}) = \varLambda_{n,r}^+(\mathbf{m})$ if $m_k \ge n$ for all $k = 1, \dots, r-1$. In the case where $m_k < n$ for some $k < r$, $\varLambda_{n,r}^+(\mathbf{m})$ is a proper subset of $\widetilde{\varLambda}_{n,r}^+(\mathbf{m})$.

In [DJM] (see also [M1] for the case where $m_k < n$ for some k), it is proven that $\mathscr{S}_{n,r}(\mathbf{m})$ is a cellular algebra with respect to the poset $(\widetilde{\varLambda}_{n,r}^+, \ge)$. For $\lambda \in \widetilde{\varLambda}_{n,r}^+(\mathbf{m})$, let $\Delta(\lambda)$ be the Weyl (cell) module corresponding to λ constructed in [DJM] (see also [M1] and [W3, Lemma 1.18]). By the general theory of cellular algebras

given in [GL], $\{\Delta(\lambda)\,|\,\lambda\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})\}$ gives a complete set of isomorphism classes of simple $\mathscr{S}_{n,r}(\mathbf{m})$-modules if $\mathscr{S}_{n,r}(\mathbf{m})$ is semi-simple. It is also proven, in [DJM], that $\mathscr{S}_{n,r}(\mathbf{m})$ is a quasi-hereditary algebra such that $\{\Delta(\lambda)\,|\,\lambda\in\Lambda^+_{n,r}(\mathbf{m})\}$ gives a complete set of standard modules if R is a field and $m_k\geq n$ for all $k=1,\dots,r-1$.

From the construction of $\Delta(\lambda)$ in [DJM], $\Delta(\lambda)$ has a basis indexed by the set of semi-standard tableaux. Since we use them in the later argument, we recall the definition of semi-standard tableaux from [DJM].

For $\lambda\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})$, the diagram $[\lambda]$ of λ is the set

$$[\lambda]=\{(i,j,k)\in\mathbb{Z}^3\,|\,1\leq i\leq m_k,\ 1\leq j\leq\lambda_i^{(k)},\ 1\leq k\leq r\}.$$

For $x=(i,j,k)\in[\lambda]$, put

$$\operatorname{res}(x)=q^{2(j-i)}Q_{k-1}.$$

For $\lambda\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})$ and $\mu\in\Lambda_{n,r}(\mathbf{m})$, a tableau of shape λ with weight μ is a map

$$T:[\lambda]\to\{(a,c)\in\mathbb{Z}\times\mathbb{Z}\,|\,a\geq 1, 1\leq c\leq r\}$$

such that $\mu_i^{(k)}=\sharp\{x\in[\lambda]\,|\,T(x)=(i,k)\}$. We define the order on $\mathbb{Z}\times\mathbb{Z}$ by $(a,c)\geq(a',c')$ if either $c>c'$, or $c=c'$ and $a\geq a'$. For a tableau T of shape λ with weight μ, we say that T is semi-standard if T satisfies the following conditions:

(i) If $T((i,j,k))=(a,c)$, then $k\leq c$,
(ii) $T((i,j,k))\leq T((i,j+1,k))$ if $(i,j+1,k)\in[\lambda]$,
(iii) $T((i,j,k))<T((i+1,j,k))$ if $(i+1,j,k)\in[\lambda]$.

For $\lambda\in\widetilde{\Lambda}^+_{n,r}(\mathbf{m})$, $\mu\in\Lambda_{n,r}(\mathbf{m})$, we denote by $\mathcal{T}_0(\lambda,\mu)$ the set of semi-standard tableaux of shape λ with weight μ. Then, from the cellular basis of $\mathscr{S}_{n,r}(\mathbf{m})$ in [DJM], we see that $\Delta(\lambda)$ has the basis

$$\{\varphi_T\,|\,T\in\mathcal{T}_0(\lambda,\mu)\text{ for some }\mu\in\Lambda_{n,r}(\mathbf{m})\}.$$

(See [DJM] for the definition of φ_T.)

§ 7. Generators of cyclotomic q-Schur algebras

In this section, we define some generators of the cyclotomic q-Schur algebra, and we obtain some relations among them which will be used to obtain the homomorphism from $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$ in the next section.

7.1. A partition λ is a non-increasing sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ of non-negative integers. For a partition $\lambda=(\lambda_1,\lambda_2,\dots)$, we denote by $\ell(\lambda)$ the length of λ which is the maximal integer l such that $\lambda_l\neq 0$. If $\sum_{i=1}^{\ell(\lambda)}\lambda_i=n$, we denote it by $\lambda\vdash n$. For a integer k and a partition $\lambda\vdash n$ such that $\ell(\lambda)\leq k$, put

$$\mathfrak{S}_k\cdot\lambda=\big\{(\mu_1,\mu_2,\dots,\mu_k)\in\mathbb{Z}^k_{\geq 0}\;\big|\;\mu_i=\lambda_{\sigma(i)},\ \sigma\in\mathfrak{S}_k\big\}.$$

7.2. For integers $t, k > 0$, we define the symmetric polynomials $\Phi^{\pm}_t(x_1, \dots, s_k) \in R[x_1, \dots, x_k]^{\mathfrak{S}_k}$ of degree t with variables $x_1, \dots, x_k$ as

$$\Phi^{\pm}_t(x_1, \dots, x_k) = \sum_{\substack{\lambda \vdash t \\ \ell(\lambda) \leq k}} (1 - q^{\mp 2})^{\ell(\lambda) - 1} \mathfrak{m}_\lambda(x_1, \dots, x_k), \tag{7.2.1}$$

where $\mathfrak{m}_\lambda(x_1, \dots, x_k) = \sum_{\mu = (\mu_1, \mu_2, \dots, \mu_k) \in \mathfrak{S}_k \cdot \lambda} x_1^{\mu_1} x_2^{\mu_2} \dots x_k^{\mu_k}$ is the monomial symmetric polynomial associated with the partition λ. For convenience, we also define

$$\Phi^{\pm}_0(x_1, \dots, x_k) = q^{\mp k \pm 1}[k]. \tag{7.2.2}$$

From the definition, we have

$$\Phi^{\pm}_1(x_1, \dots, x_k) = x_1 + x_2 + \dots + x_k \text{ and } \Phi^{\pm}_t(x_1) = x_1^t. \tag{7.2.3}$$

The polynomials $\Phi^{\pm}_t(x_1, \dots, x_k)$ satisfy the following recursive relations which will be used for calculations of some relations between generators of $\mathscr{S}_{n,r}(\mathbf{m})$ in later.

Lemma 7.3. *For $t \geq 0$, we have*

$$\begin{aligned} \Phi^{\pm}_{t+1}(x_1, \dots, x_k) &= \sum_{s=1}^{k} \Phi^{\pm}_t(x_1, \dots, x_s) x_s - q^{\mp 2} \sum_{s=1}^{k-1} \Phi^{\pm}_t(x_1, \dots, x_s) x_{s+1} \\ &= x_1^{t+1} + \sum_{s=2}^{k} \big(\Phi^{\pm}_t(x_1, \dots, x_s) x_s - q^{\mp 2} \Phi^{\pm}_t(x_1, \dots, x_{s-1}) x_s \big) \end{aligned} \tag{7.3.1}$$

and

$$\begin{aligned} &\Phi^{\pm}_{t+1}(x_1, x_2, \dots, x_k) - \Phi^{\pm}_{t+1}(x_2, \dots, x_k) \\ &= x_1 \big(\Phi^{\pm}_t(x_1, x_2, \dots, x_k) - q^{\mp 2} \Phi^{\pm}_t(x_2, \dots, x_k) \big). \end{aligned} \tag{7.3.2}$$

Proof. In the case where $t = 0$, we can check the statements by direct calculations.

Assume that $t \geq 1$. From the definition, we have

$$\begin{aligned} \Phi^{\pm}_{t+1}(x_1, \dots, x_k) &= \sum_{\substack{\lambda \vdash t+1 \\ \ell(\lambda) \leq k}} (1 - q^{\mp 2})^{\ell(\lambda) - 1} \sum_{\mu \in \mathfrak{S}_k \lambda} x_1^{\mu_1} x_2^{\mu_2} \dots x_k^{\mu_k} \\ &= \sum_{s=1}^{k} \sum_{\substack{\lambda \vdash t+1 \\ \ell(\lambda) \leq s}} (1 - q^{\mp 2})^{\ell(\lambda) - 1} \sum_{\substack{\mu \in \mathfrak{S}_s \lambda \\ \mu_s \neq 0}} x_1^{\mu_1} x_2^{\mu_2} \dots x_s^{\mu_s} \\ &= \sum_{s=1}^{k} \sum_{\substack{\lambda \vdash t+1 \\ \ell(\lambda) \leq s}} (1 - q^{\mp 2})^{\ell(\lambda) - 1} \sum_{\substack{\mu \in \mathfrak{S}_s \lambda \\ \mu_s = 1}} x_1^{\mu_1} x_2^{\mu_2} \dots x_s^{\mu_s} \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{s=1}^{k} \sum_{\substack{\lambda \vdash t+1 \\ \ell(\lambda) \le s}} (1-q^{\mp 2})^{\ell(\lambda)-1} \sum_{\substack{\mu \in \mathfrak{S}_s \lambda \\ \mu_s \ge 2}} x_1^{\mu_1} x_2^{\mu_2} \dots x_s^{\mu_s} \\
&= \sum_{s=1}^{k} \sum_{\substack{\lambda \vdash t \\ \ell(\lambda) \le s}} (1-q^{\mp 2})^{\ell(\lambda)} \sum_{\substack{\mu \in \mathfrak{S}_s \lambda \\ \mu_s = 0}} x_1^{\mu_1} x_2^{\mu_2} \dots x_{s-1}^{\mu_{s-1}} x_s \\
&\quad + \sum_{s=1}^{k} \sum_{\substack{\lambda \vdash t \\ \ell(\lambda) \le s}} (1-q^{\mp 2})^{\ell(\lambda)-1} \sum_{\substack{\mu \in \mathfrak{S}_s \lambda \\ \mu_s \ne 0}} x_1^{\mu_1} x_2^{\mu_2} \dots x_s^{\mu_s} x_s \\
&= \sum_{s=1}^{k} \Big(\sum_{\substack{\lambda \vdash t \\ \ell(\lambda) \le s}} (1-q^{\mp 2})^{\ell(\lambda)-1} \sum_{\mu \in \mathfrak{S}_s \lambda} x_1^{\mu_1} x_2^{\mu_2} \dots x_s^{\mu_s} \Big) x_s \\
&\quad - q^{\mp 2} \sum_{s=2}^{k} \Big(\sum_{\substack{\lambda \vdash t \\ \ell(\lambda) \le s}} (1-q^{\mp 2})^{\ell(\lambda)-1} \sum_{\substack{\mu \in \mathfrak{S}_s \lambda \\ \mu_s = 0}} x_1^{\mu_1} x_2^{\mu_2} \dots x_{s-1}^{\mu_{s-1}} \Big) x_s \\
&= \sum_{s=1}^{k} \Phi_t^{\pm}(x_1, \dots, x_s) x_s - q^{\mp 2} \sum_{s=1}^{k-1} \Phi_t^{\pm}(x_1, \dots, x_s) x_{s+1}.
\end{aligned}
$$

We can easily check the second equality of (7.3.1).

We prove (7.3.2) by the induction on t. In the case where $t = 1$, we can check (7.3.2) directly by using the relation (7.3.1) together with (7.2.3). Assume that $t > 1$. By (7.3.1), we have

$$
\begin{aligned}
&\Phi_{t+1}^{\pm}(x_1, x_2, \dots, x_k) - \Phi_{t+1}^{\pm}(x_2, \dots, x_k) \\
&= \Big(\sum_{s=1}^{k} \Phi_t^{\pm}(x_1, \dots, x_s) x_s - q^{\mp 2} \sum_{s=1}^{k-1} \Phi_t^{\pm}(x_1, \dots, x_s) x_{s+1} \Big) \\
&\quad - \Big(\sum_{s=2}^{k} \Phi_t^{\pm}(x_2, \dots, x_s) x_s - q^{\mp 2} \sum_{s=2}^{k-1} \Phi_t^{\pm}(x_2, \dots, x_s) x_{s+1} \Big) \\
&= \Phi_t^{\pm}(x_1) x_1 - q^{\mp 2} \Phi_t^{\pm}(x_1) x_2 + \sum_{s=2}^{k} \big(\Phi_t^{\pm}(x_1, \dots, x_s) - \Phi_t^{\pm}(x_2, \dots, x_s) \big) x_s \\
&\quad - q^{\mp 2} \sum_{s=2}^{k-1} \big(\Phi_t^{\pm}(x_1, \dots, x_s) - \Phi_t^{\pm}(x_2, \dots, x_s) \big) x_{s+1}.
\end{aligned}
$$

Applying the assumption of the induction, we have

$$
\begin{aligned}
&\Phi_{t+1}^{\pm}(x_1, x_2, \dots, x_k) - \Phi_{t+1}^{\pm}(x_2, \dots, x_k) \\
&= x_1 \Phi_{t-1}^{\pm}(x_1) x_1 - q^{\mp 2} x_1 \Phi_{t-1}^{\pm}(x_1) x_2
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{s=2}^{k} x_1\big(\Phi^{\pm}_{t-1}(x_1,x_2,\dots,x_s) - q^{\mp 2}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)\big)x_s \\
&- q^{\mp 2}\sum_{s=2}^{k-1} x_1\big(\Phi^{\pm}_{t-1}(x_1,x_2,\dots,x_s) - q^{\mp 2}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)\big)x_{s+1} \\
= x_1\Big\{&\Big(\sum_{s=1}^{k}\Phi^{\pm}_{t-1}(x_1,x_2,\dots,x_s)x_s - q^{\mp 2}\sum_{s=1}^{k-1}\Phi^{\pm}_{t-1}(x_1,x_2,\dots,x_s)x_{s+1}\Big) \\
&\quad - q^{\mp 2}\Big(\sum_{s=2}^{k}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)x_s - q^{\mp 2}\sum_{s=2}^{k-1}\Phi^{\pm}_{t-1}(x_2,\dots,x_s)x_{s+1}\Big)\Big\}.
\end{aligned}
$$

Applying the relation (7.3.1), we obtain (7.3.2). □

Remark 7.4. At first, the author defined the polynomials $\Phi^{\pm}_t(x_1,\dots,x_k)$ by using the relations (7.3.1) inductively. The definition of $\Phi^{\pm}_t(x_1,\dots,x_k)$ as in (7.2.1) was suggested by Tatsuyuki Hikita.

7.5. For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(j,l) \in \Gamma(\mathbf{m})$, put

$$
N^{\mu}_{(j,l)} = \sum_{c=1}^{l-1} |\mu^{(c)}| + \sum_{p=1}^{j} \mu^{(l)}_p .
$$

For $(j,l) \in \Gamma(\mathbf{m})$ and an integer $t \geq 0$, we define the elements $\mathcal{K}^{\pm}_{(j,l)}$ and $\mathcal{I}^{\pm}_{(j,l),t}$ of $\mathscr{S}_{(n,r)}(\mathbf{m})$ by

$$
\begin{aligned}
&\mathcal{K}^{\pm}_{(j,l)}(m_\mu) = q^{\pm \mu^{(l)}_j} m_\mu, \\
&\mathcal{I}^{+}_{(j,l),t}(m_\mu) = \begin{cases} q^{t-1} m_\mu \Phi^{+}_t(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1}) & \text{if } \mu^{(l)}_j \neq 0, \\ 0 & \text{if } \mu^{(l)}_j = 0, \end{cases} \\
&\mathcal{I}^{-}_{(j,l),t}(m_\mu) = \begin{cases} q^{-t+1} m_\mu \Phi^{-}_t(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1}) & \text{if } \mu^{(l)}_j \neq 0, \\ 0 & \text{if } \mu^{(l)}_j = 0, \end{cases}
\end{aligned}
$$

for each $\mu \in \Lambda_{n,r}(\mathbf{m})$.

It is clear that the definitions of $\mathcal{K}^{\pm}_{(j,l)}$ are well-defined. For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(j,l) \in \Gamma(\mathbf{m})$ such that $\mu^{(l)}_j \neq 0$, we see that $\Phi^{\pm}_t(L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1})$ commute with T_w for any $w \in \mathfrak{S}_\mu$ by Lemma 6.3 since $\Phi^{\pm}_t(L_{N^{\mu}_{(j,l)}}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1})$ is a symmetric polynomials with variables $L_{N^{\mu}_{(j,l)}}, L_{N^{\mu}_{(j,l)}-1}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1}$. Thus, $\Phi^{\pm}_t(L_{N^{\mu}_{(j,l)}}, \dots, L_{N^{\mu}_{(j,l)}-\mu^{(l)}_j+1})$ commute with m_μ, and the definitions of $\mathcal{I}^{\pm}_{(j,l),t}$ are well-defined.

The following lemma is immediate from definitions.

Lemma 7.6. *For $(i,k),(j,l) \in \Gamma(\mathbf{m})$ and $s,t \geq 0$, we have the following relations.*

(i) $\mathcal{K}^+_{(j,l)}\mathcal{K}^-_{(j,l)} = \mathcal{K}^-_{(j,l)}\mathcal{K}^+_{(j,l)} = 1$.
(ii) $[\mathcal{K}^+_{(i,k)}, \mathcal{K}^+_{(j,l)}] = [\mathcal{K}^+_{(i,k)}, \mathcal{I}^\sigma_{(j,l),t}] = [\mathcal{I}^\sigma_{(i,k),s}, \mathcal{I}^{\sigma'}_{(j,l),t}] = 0$ *($\sigma, \sigma' \in \{+,-\}$).*

We also have the following lemma by direct calculations.

Lemma 7.7. *For $(j,l) \in \Gamma(\mathbf{m})$, we have*

$$(\mathcal{K}^\pm_{(j,l)})^2 = 1 \pm (q-q^{-1})\mathcal{I}^\mp_{(j,l),0}.$$

7.8. For $(i,k) \in \Gamma'(\mathbf{m})$ and an integer $t \geq 0$, we define the element $\widetilde{\mathcal{K}}^\pm_{(i,k)}$ and $\mathcal{J}_{(j,l),t}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ by

$$\widetilde{\mathcal{K}}^\pm_{(i,k)} = \mathcal{K}^\pm_{(i,k)}\mathcal{K}^\mp_{(i+1,k)}$$

and

$$\mathcal{J}_{(i,k),t} = \begin{cases} \mathcal{I}^+_{(i,k),0} - \mathcal{I}^-_{(i+1,k),0} + (q-q^{-1})\mathcal{I}^+_{(i,k),0}\mathcal{I}^-_{(i+1,k),0} & \text{if } t = 0, \\ q^{-t}\mathcal{I}^+_{(i,k),t} - q^t\mathcal{I}^-_{(i+1,k),t} - (q-q^{-1})\displaystyle\sum_{b=1}^{t-1} q^{-t+2b}\mathcal{I}^+_{(i,k),t-b}\mathcal{I}^-_{(i+1,k),b} & \text{if } t > 0. \end{cases}$$

By Lemma 7.7, we have the following corollary.

Corollary 7.9. *For $(i,k) \in \Gamma'(\mathbf{m})$, we have*

$$\mathcal{J}_{(i,k),0} = \mathcal{I}^+_{(i,k),0} - (\mathcal{K}^-_{(i,k)})^2\mathcal{I}^-_{(i+1,k),0}.$$

7.10. For $N \in \mathbb{Z}_{\geq 0}$ and $\mu \in \mathbb{Z}_{>0}$, put

$$[T;N,\mu]^+ = \begin{cases} 1 + \displaystyle\sum_{h=1}^{\mu-1} q^h T_{N+1}T_{N+2}\dots T_{N+h} & \text{if } N+\mu \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

$$[T;N,\mu]^- = \begin{cases} 1 + \displaystyle\sum_{h=1}^{\mu-1} q^h T_{N-1}T_{N-2}\dots T_{N-h} & \text{if } n \geq N \geq \mu, \\ 0 & \text{otherwise.} \end{cases}$$

For convenience, we put $[T;N,0]^\pm = 0$ for any $N \in \mathbb{Z}_{\geq 0}$.

For $N, \mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, put

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = [T;N+(d-1),\mu-(d-1)]^+ \dots [T;N+1,\mu-1]^+[T;N,\mu]^+,$$
$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^- = [T;N-(d-1),\mu-(d-1)]^- \dots [T;N-1,\mu-1]^-[T;N,\mu]^-.$$

We also put $\left[\begin{smallmatrix} T;N,\mu \\ 0 \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,\mu \\ 0 \end{smallmatrix}\right]^- = 1$ for any $N, \mu \in \mathbb{Z}_{\geq 0}$.

For $N \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, put

$$(T;N,d)^+ = \begin{cases} 1 + \sum_{h=1}^{d-1} q^h T_{N+d-h} T_{N+d-(h-1)} \dots T_{N+d-2} T_{N+d-1} & \text{if } N+d \leq n, \\ 0 & \text{otherwise}, \end{cases}$$

$$(T;N,d)^- = \begin{cases} 1 + \sum_{h=1}^{d-1} q^h T_{N-d+h} T_{N-d+(h-1)} \dots T_{N-d+2} T_{N-d+1} & \text{if } n \geq N \geq d, \\ 0 & \text{otherwise}. \end{cases}$$

We also put

$$(T;N,d)^{\pm}! = (T;N,d)^{\pm}(T;N,d-1)^{\pm} \dots (T;N,1)^{\pm}.$$

The following lemma follows from Lemma 6.3 immediately.

Lemma 7.11. *For $N, \mu \in \mathbb{Z}_{\geq 0}$, we have the following.*

(i) *L_i commute with $[T;N,\mu]^+$ unless $N+\mu \geq i \geq N+1$.*
(ii) *L_i commute with $[T;N,\mu]^-$ unless $N \geq i \geq N-\mu+1$.*

Lemma 7.12. *We have the following.*

(i) *For $N, \mu \in \mathbb{Z}_{\geq 0}$ such that $N+\mu \leq n$ and $\mu \geq 3$, we have*

$$\begin{aligned}&(q^{\mu-2} T_{N+2} T_{N+3} \dots T_{N+\mu-1})(q^{\mu-1} T_{N+1} T_{N+2} \dots T_{N+\mu-1}) \\ &= (q^{\mu-1} T_{N+1} T_{N+2} \dots T_{N+\mu-1})(q^{\mu-2} T_{N+1} T_{N+2} \dots T_{N+\mu-2}).\end{aligned}$$

(ii) *For $N, \mu \in \mathbb{Z}_{\geq 0}$ such that $N \geq \mu \geq 3$, we have*

$$\begin{aligned}&(q^{\mu-2} T_{N-2} T_{N-3} \dots T_{N-\mu+1})(q^{\mu-1} T_{N-1} T_{N-2} \dots T_{N-\mu+1}) \\ &= (q^{\mu-1} T_{N-1} T_{N-2} \dots T_{N-\mu+1})(q^{\mu-2} T_{N-1} T_{N-2} \dots T_{N-\mu+2}).\end{aligned}$$

(iii) *For $N, \mu, c \in \mathbb{Z}_{\geq 0}$ such that $\mu \geq c \geq 1$, we have*

$$\begin{aligned}&[T;N+1,c]^+(q^{\mu} T_{N+1} T_{N+2} \dots T_{N+\mu}) = (q^{\mu} T_{N+1} T_{N+2} \dots T_{N+\mu})[T;N,c]^+, \\ &[T;N-1,c]^-(q^{\mu} T_{N-1} T_{N-2} \dots T_{N-\mu}) = (q^{\mu} T_{N-1} T_{N-2} \dots T_{N-\mu})[T;N,c]^-.\end{aligned}$$

Proof. (i) and (ii) follows from the defining relations of $\mathscr{H}_{n,r}$. We can prove (iii) by the induction on c. □

Lemma 7.13. *For $N, \mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, we have*

$$\begin{bmatrix} T;N,\mu \\ d \end{bmatrix}^+ = \begin{cases} (T;N,d)^+ \Big(\begin{bmatrix} T;N,d-1 \\ d-1 \end{bmatrix}^+ \\ \quad + \sum_{h=1}^{\mu-d} (q^h T_{N+d} T_{N+d+1} \dots T_{N+d+h-1}) \begin{bmatrix} T;N,d+h-1 \\ d-1 \end{bmatrix}^+ \Big) & \text{if } \mu \geq d, \\ 0 & \text{if } \mu < d, \end{cases}$$

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^- = \begin{cases} (T;N,d)^-\Big(\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^- \\ \quad + \displaystyle\sum_{h=1}^{\mu-d} (q^h T_{N-d}T_{N-d-1}\dots T_{N-d-h+1}) \left[\begin{smallmatrix} T;N,d+h-1 \\ d-1 \end{smallmatrix}\right]^- \Big) & \text{if } \mu \geq d, \\ 0 & \text{if } \mu < d. \end{cases}$$

Proof. In the case where $\mu < d$, we see that $\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^\pm = 0$ from the definitions.

First, we prove that, if $\mu > d$,

(7.13.1)
$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,\mu-1 \\ d \end{smallmatrix}\right]^+ + (T;N,d)^+(q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1}) \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+$$

by the induction on d. In the case where $d=1$, it is clear by definitions. Assume that $d>1$, then we have

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ = [T;N+(d-1),\mu-(d-1)]^+ \left[\begin{smallmatrix} T;N,\mu \\ d-1 \end{smallmatrix}\right]^+.$$

Applying the assumption of the induction, we have

$$\begin{aligned} &\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ \\ &= \Big\{[T;N+(d-1),\mu-d]^+ + (q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1})\Big\} \\ &\quad \times \Big\{ \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+ + (T;N,d-1)^+(q^{\mu-d+1}T_{N+d-1}T_{N+d}\dots T_{N+\mu-1}) \left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+ \Big\}. \end{aligned}$$

Then, by using Lemma 7.11 and Lemma 7.12, we see that

$$\begin{aligned} &\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ \\ &= [T;N+d-1,\mu-d]^+ \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+ + (q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1}) \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+ \\ &\quad + (T;N,d-1)^+(q^{\mu-d+1}T_{N+d-1}T_{N+d}\dots T_{N+\mu-1})[T;N+d-2,\mu-d]^+ \left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+ \\ &\quad + (T;N,d-1)^+(q^{\mu-d+1}T_{N+d-1}T_{N+d}\dots T_{N+\mu-1})(q^{\mu-d}T_{N+d-1}T_{N+d}\dots T_{N+\mu-2}) \\ &\qquad \times \left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+. \end{aligned}$$

Note that

$$[T;N+d-2,\mu-d]^+ + q^{\mu-d}T_{N+d-1}T_{N+d}\dots T_{N+\mu-2} = [T;N+d-2,\mu-d+1]^+$$

and $[T;N+d-2,\mu-d+1]^+ \left[\begin{smallmatrix} T;N,\mu-1 \\ d-2 \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+$, we have

$$\begin{aligned} &\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ \\ &= \left[\begin{smallmatrix} T;N,\mu-1 \\ d \end{smallmatrix}\right]^+ \\ &\quad + \big(1+(T;N,d-1)^+(qT_{N+d-1})\big)(q^{\mu-d}T_{N+d}T_{N+d+1}\dots T_{N+\mu-1}) \left[\begin{smallmatrix} T;N,\mu-1 \\ d-1 \end{smallmatrix}\right]^+. \end{aligned}$$

By definition, we see that $1+(T;N,d-1)^+(qT_{N+d-1}) = (T;N,d)^+$. Thus, we have (7.13.1).

Next, we prove that

$$\left[\begin{smallmatrix} T;N,d \\ d \end{smallmatrix}\right]^+ = (T;N,d)^+ \left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ \tag{7.13.2}$$

by the induction on d. In the case where $d=1$, it is clear from definitions. Assume that $d>1$. Note that $\left[\begin{smallmatrix} T;N,d \\ d \end{smallmatrix}\right]^+ = \left[\begin{smallmatrix} T;N,d \\ d-1 \end{smallmatrix}\right]^+$, by (7.13.1), we have

$$\begin{aligned}
\left[\begin{smallmatrix} T;N,d \\ d \end{smallmatrix}\right]^+ &= \left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ + (T;N,d-1)^+(qT_{N+d-1}) \left[\begin{smallmatrix} T;N,d-1 \\ d-2 \end{smallmatrix}\right]^+ \\
&= \big(1+(T;N,d-1)^+(qT_{N+d-1})\big) \left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ \\
&= (T;N,d)^+ \left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ .
\end{aligned}$$

Next we prove that, if $\mu \geq d$,

$$\begin{aligned}
&\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^+ \\
&= (T;N,d)^+ \Big(\left[\begin{smallmatrix} T;N,d-1 \\ d-1 \end{smallmatrix}\right]^+ + \sum_{h=1}^{\mu-d} (q^h T_{N+d} T_{N+d+1} \dots T_{N+d+h-1}) \left[\begin{smallmatrix} T;N,d+h-1 \\ d-1 \end{smallmatrix}\right]^+ \Big)
\end{aligned} \tag{7.13.3}$$

by the induction on $\mu-d$. In the case where $\mu=d$, it is just (7.13.2). Assume that $\mu>d$. By applying the assumption of the induction to the right-hand side of (7.13.1), we have (7.13.3).

It is similar for $\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^-$. □

We have the following corollary which will be used in Theorem 8.1 to consider the divided powers in cyclotomic q-Schur algebras.

Corollary 7.14. *For $N,\mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, there exist the elements $\mathfrak{H}^{\pm}(N,\mu,d) \in \mathscr{H}_{n,r}$ such that*

$$\left[\begin{smallmatrix} T;N,\mu \\ d \end{smallmatrix}\right]^{\pm} = (T;N,d)^{\pm}!\mathfrak{H}^{\pm}(N,\mu,d).$$

Proof. Note that $T_{N+d}T_{N+d+1}\dots T_{N+d+h-1}$ (resp. $T_{N-d}T_{N-d-1}\dots T_{N-d-h+1}$) commute with $(T;N,d-1)^+!$ (resp. $(T;N,d-1)^-!$), then we can prove the corollary by the induction on d using Lemma 7.13. □

7.15. For $(i,k) \in \Gamma'(\mathbf{m})$, we define the elements $\mathcal{X}^+_{(i,k),0}$ and $\mathcal{X}^-_{(i,k),0}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ by

$$\begin{aligned}
\mathcal{X}^+_{(i,k),0}(m_\mu) &= q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} [T; N^\mu_{(i,k)}, \mu^{(k)}_{i+1}]^+, \\
\mathcal{X}^-_{(i,k),0}(m_\mu) &= q^{-\mu^{(k)}_{i}+1} m_{\mu-\alpha_{(i,k)}} h^\mu_{-(i,k)} [T; N^\mu_{(i,k)}, \mu^{(k)}_{i}]^-
\end{aligned}$$

for each $\mu \in \Lambda_{n,r}(\mathbf{m})$, where we put $\mu^{(k)}_{m_k+1} = \mu^{(k+1)}_1$ if $i = m_k$, and

$$h^{\mu}_{-(i,k)} = \begin{cases} 1 & \text{if } i \neq m_k, \\ L_{N^{\mu}_{(m_k,k)}} - Q_k & \text{if } i = m_k. \end{cases}$$

Note that $m_{\mu \pm \alpha_{(i,k)}} = 0$ if $\mu \pm \alpha_{(i,k)} \notin \Lambda_{n,r}(\mathbf{m})$.

By [W1, Lemma 6.10], the definitions of $\mathcal{X}^{\pm}_{(i,k),0}$ are well-defined. (The elements $\mathcal{X}^{\pm}_{(i,k),0}$ are denoted by $\varphi^{\pm}_{(i,k)}$ in [W1].)

For $(i,k) \in \Gamma'(\mathbf{m})$ and an integer $t > 0$, we define the elements $\mathcal{X}^{\pm}_{(i,k),t}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ inductively by

$$\begin{aligned} \mathcal{X}^{+}_{(i,k),t} &= \mathcal{I}^{+}_{(i,k),1}\mathcal{X}^{+}_{(i,k),t-1} - \mathcal{X}^{+}_{(i,k),t-1}\mathcal{I}^{+}_{(i,k),1}, \\ \mathcal{X}^{-}_{(i,k),t} &= -\big(\mathcal{I}^{-}_{(i,k),1}\mathcal{X}^{-}_{(i,k),t-1} - \mathcal{X}^{-}_{(i,k),t-1}\mathcal{I}^{-}_{(i,k),1}\big). \end{aligned} \tag{7.15.1}$$

Lemma 7.16. *For $(i,k) \in \Gamma'(\mathbf{m})$, $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$, we have*

$$\mathcal{K}^{+}_{(j,l)}\mathcal{X}^{\pm}_{(i,k),t}\mathcal{K}^{-}_{(j,l)} = q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^{\pm}_{(i,k),t}.$$

Proof. We see the statement in the case where $t = 0$ from the definitions directly. Then we can prove the statement by the induction on t using (7.15.1) together with Lemma 7.6. □

We can describe the elements $\mathcal{X}^{\pm}_{(i,k),t}$ of $\mathscr{S}_{n,r}(\mathbf{m})$ precisely as follows.

Lemma 7.17. *For $(i,k) \in \Gamma'(\mathbf{m})$, $t \geq 0$ and $\mu \in \Lambda_{n,r}(\mathbf{m})$, we have the followings.*

(i) $\mathcal{X}^{+}_{(i,k),t}(m_\mu) = q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} L^{t}_{N^{\mu}_{(i,k)}+1}[T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}]^{+}$.

(ii) $\mathcal{X}^{-}_{(i,k),t}(m_\mu) = q^{-\mu^{(k)}_{i}+1} m_{\mu-\alpha_{(i,k)}} L^{t}_{N^{\mu}_{(i,k)}} h^{\mu}_{-(i,k)}[T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i}]^{-}$.

Proof. We prove (i). We can easily show that $\mathcal{X}^{+}_{(i,k),t}(m_\mu) = 0$ if $\mu^{(k)}_{i+1} = 0$ by the induction on t using (7.15.1). Assume that $\mu^{(k)}_{i+1} \neq 0$. If $t = 0$, then it is just the definition of $\mathcal{X}^{+}_{(i,k),0}$. We prove the equation for $t > 0$ by the induction on t. Note that $(\mu + \alpha_{(i,k)})^{(k)}_i = \mu^{(k)}_i + 1$ and $N^{\mu+\alpha_{(i,k)}}_{(i,k)} = N^{\mu}_{(i,k)} + 1$, by the assumption of the induction, we have

$$\begin{aligned} \mathcal{I}^{+}_{(i,k),1}\mathcal{X}^{+}_{(i,k),t-1}(m_\mu) =& q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} \\ & \times (L_{N^{\mu}_{(i,k)}+1} + L_{N^{\mu}_{(i,k)}} + L_{N^{\mu}_{(i,k)}-1} + \cdots + L_{N^{\mu}_{(i,k)}-\mu^{(k)}_i+1}) \\ & \times L^{t-1}_{N^{\mu}_{(i,k)}+1}[T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}]^{+}. \end{aligned}$$

On the other hand, we have

$$\mathcal{X}^{+}_{(i,k),t-1}\mathcal{I}^{+}_{(i,k),1}(m_\mu) = \delta_{(\mu^{(k)}_i \neq 0)} q^{-\mu^{(k)}_{i+1}+1} m_{\mu+\alpha_{(i,k)}} L^{t-1}_{N^{\mu}_{(i,k)}+1}[T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}]^{+}$$

$$\times (L_{N^{\mu}_{(i,k)}} + L_{N^{\mu}_{(i,k)}-1} + \cdots + L_{N^{\mu}_{(i,k)}-\mu_i^{(k)}+1}).$$

Thus, by (7.15.1) and Lemma 7.11, we have (i). (ii) is similar. □

Proposition 7.18. *For $(i,k),(j,l) \in \Gamma'(\mathbf{m})$ and $s,t \geq 0$, we have the following relations.*

(i) $[\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{X}^{\pm}_{(j,l),s}] = 0$ *if* $(j,l) \neq (i,k),(i\pm 1,k)$.
(ii) $\mathcal{X}^{\pm}_{(i,k),t+1}\mathcal{X}^{\pm}_{(i,k),s} - q^{\pm 2}\mathcal{X}^{\pm}_{(i,k),s}\mathcal{X}^{\pm}_{(i,k),t+1} = q^{\pm 2}\mathcal{X}^{\pm}_{(i,k),t}\mathcal{X}^{\pm}_{(i,k),s+1} - \mathcal{X}^{\pm}_{(i,k),s+1}\mathcal{X}^{\pm}_{(i,k),t}$.
(iii)

$$\mathcal{X}^{+}_{(i,k),t+1}\mathcal{X}^{+}_{(i+1,k),s} - q^{-1}\mathcal{X}^{+}_{(i+1,k),s}\mathcal{X}^{+}_{(i,k),t+1} = \mathcal{X}^{+}_{(i,k),t}\mathcal{X}^{+}_{(i+1,k),s+1} - q\mathcal{X}^{+}_{(i+1,k),s+1}\mathcal{X}^{+}_{(i,k),t},$$
$$\mathcal{X}^{-}_{(i+1,k),s}\mathcal{X}^{-}_{(i,k),t+1} - q^{-1}\mathcal{X}^{-}_{(i,k),t+1}\mathcal{X}^{-}_{(i+1,k),s} = \mathcal{X}^{-}_{(i+1,k),s+1}\mathcal{X}^{-}_{(i,k),t} - q\mathcal{X}^{-}_{(i,k),t}\mathcal{X}^{-}_{(i+1,k),s+1}.$$

Proof. (i) follows from Lemma 7.17 using Lemma 6.3.

We prove (ii). We may assume that $t \geq s$ by multiplying -1 to both sides if necessary. We prove

$$\mathcal{X}^{+}_{(i,k),t+1}\mathcal{X}^{+}_{(i,k),s} - q^{2}\mathcal{X}^{+}_{(i,k),s}\mathcal{X}^{+}_{(i,k),t+1} = q^{2}\mathcal{X}^{+}_{(i,k),t}\mathcal{X}^{+}_{(i,k),s+1} - \mathcal{X}^{+}_{(i,k),s+1}\mathcal{X}^{+}_{(i,k),t}. \tag{7.18.1}$$

Put $N = N^{\mu}_{(i,k)}$. By Lemma 7.17 together with Lemma 7.11, for $\mu \in \Lambda_{n,r}(\mathbf{m})$, we have

$$\begin{aligned}&\mathcal{X}^{+}_{(i,k),t+1}\mathcal{X}^{+}_{(i,k),s}(m_\mu)\\ &= q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} L^{s}_{N+1}L^{t+1}_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^{+}[T;N,\mu^{(k)}_{i+1}]^{+}.\end{aligned} \tag{7.18.2}$$

Thus, we may assume that $\mu^{(k)}_{i+1} \geq 2$ since $m_{\mu+2\alpha_{(i,k)}} = 0$ if $\mu^{(k)}_{i+1} < 2$. By the induction on $\mu^{(k)}_{i+1}$, we can show that

(7.18.3)
$$T_{N+1}[T;N+1,\mu^{(k)}_{i+1}-1]^{+}[T;N,\mu^{(k)}_{i+1}]^{+} = q[T;N+1,\mu^{(k)}_{i+1}-1]^{+}[T;N,\mu^{(k)}_{i+1}]^{+}.$$

We also have, by Lemma 6.3,

(7.18.4)
$$\begin{aligned}L^{s}_{N+1}L^{t+1}_{N+2} &= (L_{N+1}L_{N+2})^{s}(T_{N+1}L_{N+1}T_{N+1})L^{t-s}_{N+2}\\ &= T_{N+1}(L_{N+1}L_{N+2})^{s}L_{N+1}\big\{L^{t-s}_{N+1}T_{N+1} + (q-q^{-1})\sum_{p=1}^{t-s}L^{t-s-p}_{N+1}L^{p}_{N+2}\big\}\\ &= T_{N+1}L^{t+1}_{N+1}L^{s}_{N+2}T_{N+1} + (q-q^{-1})T_{N+1}\sum_{p=1}^{t-s}L^{t-p+1}_{N+1}L^{s+p}_{N+2}.\end{aligned}$$

Then, (7.18.2) by using (6.4.2), (7.18.3) and (7.18.4), we have

$$
\begin{aligned}
&\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i,k),s}(m_\mu)\\
&= q^2 q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} L^{t+1}_{N+1} L^s_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+\\
&\quad + q(q-q^{-1})q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L^{t-p+1}_{N+1} L^{s+p}_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+\\
&= q^2 \mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t+1}(m_\mu)\\
&\quad + q(q-q^{-1})q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L^{t-p+1}_{N+1} L^{s+p}_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&q^2\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s+1}(m_\mu)\\
&= q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} T_{N+1} L^{s+1}_{N+1} L^t_{N+2} T_{N+1}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+\\
&= q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} L^{t}_{N+1} L^{s+1}_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+\\
&\quad + q(q-q^{-1})q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L^{t-p+1}_{N+1} L^{s+p}_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+\\
&= \mathcal{X}^+_{(i,k),s+1}\mathcal{X}^+_{(i,k),t}(m_\mu)\\
&\quad + q(q-q^{-1})q^{-2\mu^{(k)}_{i+1}+3} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L^{t-p+1}_{N+1} L^{s+p}_{N+2}[T;N+1,\mu^{(k)}_{i+1}-1]^+[T;N,\mu^{(k)}_{i+1}]^+.
\end{aligned}
$$

Thus, we have (7.18.1). Another case of (ii) is proven in a similar way.

We prove (iii). Put $N = N^\mu_{(i,k)}$. In the case where $\mu^{(k)}_{i+1} = 0$, by Lemma 7.17 together with Lemma 7.11, we see that

$$
\begin{aligned}
&(\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1})(m_\mu)\\
&= (\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t})(m_\mu)\\
&= q^{-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{s+t+1}_{N+1}[T;N,\mu^{(k)}_{i+2}]^+.
\end{aligned}
\tag{7.18.5}
$$

Assume that $\mu^{(k)}_{i+1} \neq 0$. By Lemma 7.17 together with Lemma 7.11, we have

(7.18.6)

$$
\begin{aligned}
&(\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1})(m_\mu)\\
&= q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{t+1}_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L^s_{N+\mu^{(k)}_{i+1}+1}\\
&\qquad \times [T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+
\end{aligned}
$$

and

(7.18.7)
$$
\begin{aligned}
&(\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t})(m_\mu) \\
&= -(q-q^{-1})q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+2} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1} L^{s+1}_{N+\mu^{(k)}_{i+1}+1} \\
&\qquad \times [T; N, \mu^{(k)}_{i+1}]^+ [T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+ \\
&\quad + q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L^{s+1}_{N+\mu^{(k)}_{i+1}+1} \\
&\qquad \times [T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+.
\end{aligned}
$$

By the induction on $\mu^{(k)}_{i+1}$ using Lemma 6.3, we can prove that

$$
\begin{aligned}
&(T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L_{N+\mu^{(k)}_{i+1}+1} \\
&= L_{N+1}(T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) + \delta_{(\mu^{(k)}_{i+1}\geq 2)}(q-q^{-1})L_{N+2}(T_{N+2}T_{N+3}\dots T_{N+\mu^{(k)}_{i+1}}) \\
&\quad + (q-q^{-1}) \sum_{p=1}^{\mu^{(k)}_{i+1}-2} (T_{N+1}T_{N+2}\dots T_{N+p}) L_{N+p+2}(T_{N+p+2}T_{N+p+3}\dots T_{N+\mu^{(k)}_{i+1}}) \\
&\quad + (q-q^{-1})(T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}-1}) L_{N+\mu^{(k)}_{i+1}+1}.
\end{aligned}
$$

By using Lemma 6.3 and (6.4.2), this equation implies

$$
\begin{aligned}
&m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L_{N+\mu^{(k)}_{i+1}+1} \\
&= m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{t+1}_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) \\
&\quad + q(q-q^{-1}) m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^t_{N+1}[T; N, \mu^{(k)}_{i+1}]^+ L_{N+\mu^{(k)}_{i+1}+1}.
\end{aligned}
\tag{7.18.8}
$$

Thus, (7.18.7) and (7.18.8) imply

(7.18.9)
$$
\begin{aligned}
&(\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t})(m_\mu) \\
&= q^{-\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+1} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L^{t+1}_{N+1}(q^{\mu^{(k)}_{i+1}} T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}}) L^s_{N+\mu^{(k)}_{i+1}+1} \\
&\qquad \times [T; N+\mu^{(k)}_{i+1}, \mu^{(k)}_{i+2}]^+.
\end{aligned}
$$

By (7.18.5), (7.18.6) and (7.18.9), we have

$$
\mathcal{X}^+_{(i,k),t+1}\mathcal{X}^+_{(i+1,k),s} - q^{-1}\mathcal{X}^+_{(i+1,k),s}\mathcal{X}^+_{(i,k),t+1} = \mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),s+1} - q\mathcal{X}^+_{(i+1,k),s+1}\mathcal{X}^+_{(i,k),t}.
$$

Another case of (iii) is proven in a similar way. □

Proposition 7.19. *For $(i,k) \in \Gamma'(\mathbf{m})$ and $s,t,u \geq 0$, we have the followings.*

(i)

$$\mathcal{X}^+_{(i\pm1,k),u}\big(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s}\big)+\big(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s}\big)\mathcal{X}^+_{(i\pm1,k),u}$$
$$=(q+q^{-1})\big(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i\pm1,k),u}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i\pm1,k),u}\mathcal{X}^+_{(i,k),s}\big).$$

(ii)

$$\mathcal{X}^-_{(i\pm1,k),u}\big(\mathcal{X}^-_{(i,k),s}\mathcal{X}^-_{(i,k),t}+\mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i,k),s}\big)+\big(\mathcal{X}^-_{(i,k),s}\mathcal{X}^-_{(i,k),t}+\mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i,k),s}\big)\mathcal{X}^-_{(i\pm1,k),u}$$
$$=(q+q^{-1})\big(\mathcal{X}^-_{(i,k),s}\mathcal{X}^-_{(i\pm1,k),u}\mathcal{X}^-_{(i,k),t}+\mathcal{X}^-_{(i,k),t}\mathcal{X}^-_{(i\pm1,k),u}\mathcal{X}^-_{(i,k),s}\big).$$

Proof. By Lemma 7.17 together with Lemma 7.11, we have

(7.19.1)
$$\begin{aligned}
&(\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}-q\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t})(m_\mu)\\
&=-\delta_{(\mu^{(k)}_{i+1}=1)}q^{-\mu^{(k)}_{i+2}+2}m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}L^t_{N+1}L^{s+u}_{N+2}[T;N+1,\mu^{(k)}_{i+2}]^+\\
&\quad-\delta_{(\mu^{(k)}_{i+1}\geq2)}q^{-2\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+4}m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}L^t_{N+1}L^s_{N+2}\\
&\qquad\times(q^{\mu^{(k)}_{i+1}-1}T_{N+2}T_{N+3}\dots T_{N+\mu^{(k)}_{i+1}})[T;N,\mu^{(k)}_{i+1}]^+L^u_{N+\mu^{(k)}_{i+1}+1}[T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+
\end{aligned}$$

and

$$\begin{aligned}
&(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),u}-q^{-1}\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t})(m_\mu)\\
&=q^{-2\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+2}m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}L^t_{N+1}L^s_{N+2}\\
&\quad\times[T;N+1,\mu^{(k)}_{i+1}]^+(q^{\mu^{(k)}_{i+1}}T_{N+1}T_{N+2}\dots T_{N+\mu^{(k)}_{i+1}})L^u_{N+\mu^{(k)}_{i+1}+1}[T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+.
\end{aligned}$$

Applying Lemma 7.12 (iii), we have

(7.19.2)
$$\begin{aligned}
&(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),u}-q^{-1}\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t})(m_\mu)\\
&=\delta_{(\mu^{(k)}_{i+1}=1)}q^{-\mu^{(k)}_{i+2}+1}m_{\mu+2\alpha_{(i,k)}+\alpha_{i+1,k)}}L^t_{N+1}L^s_{N+2}T_{N+1}L^u_{N+2}[T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+\\
&\quad+\delta_{\mu^{(k)}_{i+1}\geq2}q^{-2\mu^{(k)}_{i+1}-\mu^{(k)}_{i+2}+3}m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}L^t_{N+1}L^s_{N+2}T_{N+1}\\
&\qquad\times(q^{\mu^{(k)}_{i+1}-1}T_{N+2}T_{N+3}\dots T_{N+\mu^{(k)}_{i+1}})[T;N;\mu^{(k)}_{i+1}]^+L^u_{N+\mu^{(k)}_{i+1}+1}[T;N+\mu^{(k)}_{i+1},\mu^{(k)}_{i+2}]^+.
\end{aligned}$$

We see that

$$\begin{aligned}
&m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}(L^t_{N+1}L^s_{N+2}+L^s_{N+1}L^t_{N+2})T_{N+1}\\
&=m_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}T_{N+1}(L^t_{N+1}L^s_{N+2}+L^s_{N+1}L^t_{N+2})\\
&=qm_{\mu+2\alpha_{(i,k)}+\alpha_{(i+1,k)}}(L^t_{N+1}L^s_{N+2}+L^s_{N+1}L^t_{N+2})
\end{aligned}$$

by Lemma 6.3 and (6.4.2). Then (7.19.1) and (7.19.2) imply

$$\begin{aligned}&\mathcal{X}^+_{(i+1,k),u}(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s})+(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i,k),s})\mathcal{X}^+_{(i+1,k),u}\\&=(q+q^{-1})(\mathcal{X}^+_{(i,k),s}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),t}+\mathcal{X}^+_{(i,k),t}\mathcal{X}^+_{(i+1,k),u}\mathcal{X}^+_{(i,k),s}).\end{aligned}$$

The other cases of the proposition are proven in a similar way. □

By direct calculations, we have the following lemma.

Lemma 7.20. *For $(i,k)\in\Gamma'(\mathbf{m})$, $(j,l)\in\Gamma(\mathbf{m})$, $t\geq 0$, we have the followings.*

(i) $q^{\pm a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),0}\mathcal{X}^+_{(i,k),t}-q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^+_{(i,k),t}\mathcal{I}^\pm_{(j,l),0}=a_{(i,k)(j,l)}\mathcal{X}^+_{(i,k),t}$.

(ii) $q^{\mp a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),0}\mathcal{X}^-_{(i,k),t}-q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^-_{(i,k),t}\mathcal{I}^\pm_{(j,l),0}=-a_{(i,k)(j,l)}\mathcal{X}^-_{(i,k),t}$.

We also have the following proposition.

Proposition 7.21. *For $(i,k)\in\Gamma'(\mathbf{m})$, $(j,l)\in\Gamma(\mathbf{m})$, $s,t\geq 0$, we have the followings.*

(i) $[\mathcal{I}^\pm_{(j,l),s+1},\mathcal{X}^+_{(i,k),t}]=q^{\pm a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),s}\mathcal{X}^+_{(i,k),t+1}-q^{\mp a_{(i,k)(j,l)}}\mathcal{X}^+_{(i,k),t+1}\mathcal{I}^\pm_{(j,l),s}$.

(ii) $[\mathcal{I}^\pm_{(j,l),s+1},\mathcal{X}^-_{(i,k),t}]=q^{\mp a_{(i,k)(j,l)}}\mathcal{I}^\pm_{(j,l),s}\mathcal{X}^-_{(i,k),t+1}-q^{\pm a_{(i,k)(j,l)}}\mathcal{X}^-_{(i,k),t+1}\mathcal{I}^\pm_{(j,l),s}$.

Proof. By Lemma 7.17 together with Lemma 6.3, we see that

$$[\mathcal{I}^\sigma_{(j,l),s},\mathcal{X}^{\sigma'}_{(i,k),t}]=0 \text{ if } (j,l)\neq(i,k),(i+1,k),$$

where $\sigma,\sigma'\in\{+,-\}$. Thus, it is enough to prove the cases where $(j,l)=(i,k)$ or $(j,l)=(i+1,k)$. We prove

$$[\mathcal{I}^+_{(i,k),s+1},\mathcal{X}^+_{(i,k),t}]=q\mathcal{I}^+_{(i,k),s}\mathcal{X}^+_{(i,k),t+1}-q^{-1}\mathcal{X}^+_{(i,k),t+1}\mathcal{I}^+_{(i,k),s}. \tag{7.21.1}$$

For $\mu\in\Lambda_{n,r}(\mathbf{m})$, put $N=N^\mu_{(i,k)}$. Then, by Lemma 7.17 together with Lemma 7.11, we have

$$\begin{aligned}&(\mathcal{I}^+_{(i,k),s+1}\mathcal{X}^+_{(i,k),t}-\mathcal{X}^+_{(i,k),t}\mathcal{I}^+_{(i,k),s+1})(m_\mu)\\&=q^{s-\mu^{(k)}_{i+1}+1}m_{\mu+\alpha_{(i,k)}}\\&\quad\times\big(\Phi^+_{s+1}(L_{N+1},L_N,L_{N-1},\dots,L_{N-\mu^{(k)}_i+1})-\Phi^+_{s+1}(L_N,L_{N-1},\dots,L_{N-\mu^{(k)}_i+1})\big)\\&\quad\times L^t_{N+1}[T;N,\mu^{(k)}_{i+1}]^+.\end{aligned}$$

By (7.3.2), we have

$$\begin{aligned}&(\mathcal{I}^+_{(i,k),s+1}\mathcal{X}^+_{(i,k),t}-\mathcal{X}^+_{(i,k),t}\mathcal{I}^+_{(i,k),s+1})(m_\mu)\\&=q^{s-\mu^{(k)}_{i+1}+1}m_{\mu+\alpha_{(i,k)}}\\&\quad\times L_{N+1}\big(\Phi^+_s(L_{N+1},L_N,\dots,L_{N-\mu^{(k)}_i+1})-q^{-2}\Phi^+_s(L_N,L_{N-1},\dots,L_{N-\mu^{(k)}_i+1})\big)\\&\quad\times L^t_{N+1}[T;N,\mu^{(k)}_{i+1}]^+\end{aligned}$$

$$
\begin{aligned}
&= q^{(s-1)-\mu_{i+1}^{(k)}+1} m_{\mu+\alpha_{(i,k)}} \\
&\quad \times \Big\{ q\Phi_s^+(L_{N+1}, L_N, \dots, L_{N-\mu_i^{(k)}+1}) L_{N+1}^{t+1}[T; N, \mu_{i+1}^{(k)}]^+ \\
&\qquad - q^{-1} L_{N+1}^{t+1}[T; N, \mu_{i+1}^{(k)}]^+ \Phi_s^+(L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \Big\} \\
&= (q\mathcal{I}_{(i,k),s}^+ \mathcal{X}_{(i,k),t+1}^+ - q^{-1}\mathcal{X}_{(i,k),t+1}^+ \mathcal{I}_{(i,k),s}^+)(m_\mu).
\end{aligned}
$$

Now we proved (7.21.1). Other cases are proven in a similar way. □

Proposition 7.22. *For $(i,k),(j,l) \in \Gamma'(\mathbf{m})$ such that $(i,k) \neq (j,l)$ and $s,t \geq 0$, we have*

$$
[\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(j,l),s}^-] = 0.
$$

Proof. By Lemma 7.17, for $\mu \in \Lambda_{n,r}(\mathbf{m})$, we have

$$
\begin{aligned}
&\mathcal{X}_{(i,k),t}^+ \mathcal{X}_{(j,l),s}^-(m_\mu) \\
&= q^{-\mu_j^{(l)} - (\mu-\alpha_{(j,l)})_{i+1}^{(k)} + 2} m_{\mu+\alpha_{(i,k)}-\alpha_{(j,l)}} \\
&\quad \times L^t_{N_{(i,k)}^{\mu-\alpha_{(j,l)}}+1} [T; N_{(i,k)}^{\mu-\alpha_{(j,l)}}, (\mu-\alpha_{(j,l)})_{i+1}^{(k)}]^+ L^s_{N_{(j,l)}^\mu} h_{-(j,l)}^\mu [T; N_{(j,l)}^\mu, \mu_j^{(l)}]^-
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{X}_{(j,l),s}^- \mathcal{X}_{(i,k),t}^+(m_\mu) \\
&= q^{-\mu_{i+1}^{(k)} - (\mu+\alpha_{(i,k)})_j^{(l)} + 2} m_{\mu+\alpha_{(i,k)}-\alpha_{(j,l)}} \\
&\quad \times L^s_{N_{(j,l)}^{\mu+\alpha_{(i,k)}}} h_{-(j,l)}^{\mu+\alpha_{(i,k)}} [T; N_{(j,l)}^{\mu+\alpha_{(i,k)}}, (\mu+\alpha_{(i,k)})_j^{(l)}]^- L^t_{N_{(i,k)}^\mu+1} [T; N_{(i,k)}^\mu, \mu_{i+1}^{(k)}]^+.
\end{aligned}
$$

Since $(i,k) \neq (j,l)$, we have

$$
\begin{aligned}
&N_{(i,k)}^\mu = N_{(i,k)}^{\mu-\alpha_{(j,l)}}, \quad N_{(j,l)}^\mu = N_{(j,l)}^{\mu+\alpha_{(i,k)}}, \\
&(\mu-\alpha_{(j,l)})_{i+1}^{(k)} = \begin{cases} \mu_{i+1}^{(k)} & \text{if } (j,l) \neq (i+1,k), \\ \mu_{i+1}^{(k)} - 1 & \text{if } (j,l) = (i+1,k), \end{cases} \\
&(\mu+\alpha_{(i,k)})_j^{(l)} = \begin{cases} \mu_j^{(l)} & \text{if } (j,l) \neq (i+1,k), \\ \mu_j^{(l)} - 1 & \text{if } (j,l) = (i+1,k). \end{cases} \\
&h_{-(j,l)}^\mu = h_{-(j,l)}^{\mu+\alpha_{(i,k)}} = \begin{cases} 1 & \text{if } j \neq m_j, \\ L_{N_{(m_l,l)}^\mu} - Q_l & \text{if } j = m_l. \end{cases}
\end{aligned}
$$

Then, by Lemma 7.11, we have

$$
[T; N_{(i,k)}^{\mu-\alpha_{(j,l)}}, (\mu-\alpha_{(j,l)})_{i+1}^{(k)}]^+ L^s_{N_{(j,l)}^\mu} h_{-(j,l)}^\mu = L^s_{N_{(j,l)}^\mu} h_{-(j,l)}^\mu [T; N_{(i,k)}^{\mu-\alpha_{(j,l)}}, (\mu-\alpha_{(j,l)})_{i+1}^{(k)}]^+
$$

and

$$[T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_j]^- L^t_{N^\mu_{(i,k)}+1} = L^t_{N^\mu_{(i,k)}+1}[T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_j]^-.$$

Thus, in order to prove the proposition, it is enough to show that

$$\begin{aligned}[T; N^{\mu-\alpha_{(j,l)}}_{(i,k)}, (\mu-\alpha_{(j,l)})^{(k)}_{i+1}]^+ [T; N^\mu_{(j,l)}, \mu^{(l)}_j]^- \\ = [T; N^{\mu+\alpha_{(i,k)}}_{(j,l)}, (\mu+\alpha_{(i,k)})^{(l)}_j]^- [T; N^\mu_{(i,k)}, \mu^{(k)}_{i+1}]^+.\end{aligned} \tag{7.22.1}$$

If $(j,l) \neq (i+1,k)$, we see easily that (7.22.1) holds since the product is cummutative in each side. In the case where $(j,l) = (i+1,k)$, we can prove that (7.22.1) by the induction on $\mu^{(k)}_{i+1}$. Now we proved the proposition. □

Remark 7.23. There is an error in the proof of [W1, Proposition 6.11 (i)] (see the case where $(j,l) = (i+1,k)$). The above proof also gives a fixed proof of [W1, Proposition 6.11 (i)] as a special case.

We prepare some technical lemmas.

Lemma 7.24. *For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma(\mathbf{m})$, we have the followings.*

(i) *For $t \geq 0$ and $1 \leq p \leq \mu^{(k)}_i$, we have*

$$m_\mu L^t_{N^\mu_{(i,k)}}[T; N^\mu_{(i,k)}, p]^- = q^{2p-2} m_\mu \Phi^+_t(L_{N^\mu_{(i,k)}}, L_{N^\mu_{(i,k)}-1}, \dots, L_{N^\mu_{(i,k)}-p+1}).$$

(ii) *For $t \geq 0$ and $1 \leq p \leq \mu^{(k)}_{i+1}$, we have*

$$m_\mu L^t_{N^\mu_{(i,k)}+1}[T; N^\mu_{(i,k)}, p]^+ = m_\mu \Phi^-_t(L_{N^\mu_{(i,k)}+1}, L_{N^\mu_{(i,k)}+2}, \dots, L_{N^\mu_{(i,k)}+p}).$$

Proof. In the case where $t = 0$, we have (i) and (ii) from (6.4.2).

We prove (i) for $t > 0$. Put $N = N^\mu_{(i,k)}$. For $1 \leq h \leq \mu^{(k)}_i - 1$, by the induction on h together with Lemma 6.3 and (6.4.2), we can show that

(7.24.1)

$$\begin{aligned} & m_\mu L^t_N (T_{N-1}T_{N-2}\dots T_{N-h}) \\ &= m_\mu \big\{ (q-q^{-1})q^{h-1}L^t_N + \sum_{s=2}^{h} (q-q^{-1})q^{h-s} L^{t-1}_N (T_{N-1}T_{N-2}\dots T_{N-s+1}) L_{N-s+1} \\ &\qquad + L^{t-1}_N (T_{N-1}T_{N-2}\dots T_{N-h}) L_{N-h} \big\}. \end{aligned}$$

We prove that

$$\begin{aligned} & m_\mu L^t_N (T_{N-1}T_{N-2}\dots T_{N-h}) \\ &= m_\mu \big(q^h \Phi^+_t(L_N, L_{N-1}, \dots, L_{N-h}) - q^{h-2}\Phi^+_t(L_N, L_{N-1}, \dots, L_{N-h+1}) \big) \end{aligned} \tag{7.24.2}$$

by the induction on t. In the case where $t=1$, by (7.24.1) together with (6.4.2), we have

$$
\begin{aligned}
& m_\mu L_N(T_{N-1}T_{N-2}\dots T_{N-h}) \\
&= m_\mu\big\{(q-q^{-1})q^{h-1}L_N + \sum_{s=2}^{h}(q-q^{-1})q^{h-s}q^{s-1}L_{N-s+1} + q^h L_{N-h}\big\} \\
&= m_\mu\big(q^h\Phi_1^+(L_N, L_{N-1},\dots,L_{N-h}) - q^{h-2}\Phi_1^+(L_N,L_{N-1},\dots,L_{N-h+1})\big).
\end{aligned}
$$

Assume that $t>1$. Applying the assumption of the induction to (7.24.1), we have

$$
\begin{aligned}
& m_\mu L_N^t(T_{N-1}T_{N-2}\dots T_{N-h}) \\
&= m_\mu\big\{(q-q^{-1})q^{h-1}L_N^t \\
&\qquad + \sum_{s=2}^{h}(q-q^{-1})q^{h-s}\big(q^{s-1}\Phi_{t-1}^+(L_N,L_{N-1},\dots,L_{N-s+1}) \\
&\qquad\qquad - q^{s-3}\Phi_{t-1}^+(L_N,L_{N-1},\dots,L_{N-s+2})\big)L_{N-s+1} \\
&\qquad + \big(q^h\Phi_{t-1}^+(L_N,L_{N-1},\dots,L_{N-h}) - q^{h-2}\Phi_{t-1}^+(L_N,L_{N-1},\dots,L_{N-h+1})\big)L_{N-h}\big\}
\end{aligned}
$$

Put $s'=s-1$, we have

$$
\begin{aligned}
& m_\mu L_N^t(T_{N-1}T_{N-2}\dots T_{N-h}) \\
&= m_\mu\Big\{q^h\Big(L_N^t + \sum_{s=1}^{h}\big(\Phi_{t-1}^+(L_N,L_{N-1},\dots,L_{N-s'})L_{N-s'} \\
&\qquad\qquad - q^{-2}\Phi_{t-1}(L_N,L_{N-1},\dots,L_{N-s'+1})L_{N-s'}\big)\Big) \\
&\qquad - q^{h-2}\Big(L_N^t + \sum_{s=1}^{h-1}\big(\Phi_{t-1}^+(L_N,L_{N-1},\dots,L_{N-s'})L_{N-s'} \\
&\qquad\qquad - q^{-2}\Phi_{t-1}(L_N,L_{N-1},\dots,L_{N-s'+1})L_{N-s'}\big)\Big)\Big\}.
\end{aligned}
$$

Applying (7.3.1) to the right-hand side, we have (7.24.2). Thanks to (7.24.2), we have

$$
\begin{aligned}
& m_\mu L_N^t[T;N,p]^- \\
&= m_\mu L_N^t(1+\sum_{h=1}^{p-1}q^h T_{N-1}T_{N-2}\dots T_{N-h}) \\
&= m_\mu\big\{\Phi_t^+(L_N) + \sum_{h=1}^{p-1}\big(q^{2h}\Phi_t^+(L_N,L_{N-1},\dots,L_{N-h}) \\
&\qquad\qquad - q^{2h-2}\Phi_t^+(L_N,L_{N-1},\dots,L_{N-h+1})\big)\big\}
\end{aligned}
$$

$$= q^{2p-2} m_\mu \Phi^+_t(L_N, L_{N-1}, \dots, L_{N-p}).$$

Now we obtained (i).

For $t > 0$ and $1 \le h \le \mu^{(k)}_{i+1} - 1$, by the induction on h using Lemma 6.3 and (6.4.2), we can show that

$$\begin{aligned} (7.24.3)\qquad & m_\mu L^t_{N+1}(T_{N+1}T_{N+2}\dots T_{N+h}) \\ &= q^{-h} m_\mu L^{t-1}_{N+1}\big\{(1-q^2)\big(1+\sum_{s=1}^{h-1} q^s T_{N+1}T_{N+2}\dots T_{N+s}\big) \\ &\qquad + q^h T_{N+1}T_{N+2}\dots T_{N+h}\big\} L_{N+h+1}. \end{aligned}$$

We prove (ii) by the induction on t. We have already proved (ii) in the case where $t = 0$.

Assume that $t > 0$. By (7.24.3), we have

$$\begin{aligned} & m_\mu L^t_{N+1}[T; N, p]^+ \\ &= m_\mu L^t_{N+1}\big(1 + \sum_{h=1}^{p-1} q^h T_{N+1}T_{N+2}\dots T_{N+h}\big) \\ &= m_\mu L^{t-1}_{N+1}\Big\{L_{N+1} + \sum_{h=1}^{p-1}\big\{(1-q^2)(1+\sum_{s=1}^{h-1} q^s T_{N+1}T_{N+2}\dots T_{N+s}) \\ &\qquad + q^h T_{N+1}T_{N+2}\dots T_{N+h}\big\} L_{N+h+1}\Big\} \\ &= m_\mu L^{t-1}_{N+1}\Big\{\sum_{h=1}^{p}[T; N, h]^+ L_{N+h} - q^2 \sum_{h=1}^{p-1}[T; N, h]^+ L_{N+h+1}\Big\}. \end{aligned}$$

Applying the assumption of the induction, we have

$$\begin{aligned} & m_\mu L^t_{N+1}[T; N, p]^+ \\ &= m_\mu\Big\{\sum_{h=1}^{p} \Phi^-_{t-1}(L_{N+1}, L_{N+2}, \dots, L_{N+h}) L_{N+h} \\ &\qquad - q^2 \sum_{h=1}^{p-1} \Phi^-_{t-1}(L_{N+1}, L_{N+2}, \dots, L_{N+h}) L_{N+h+1}\Big\}. \end{aligned}$$

Applying (7.3.1), we have

$$m_\mu L^t_{N+1}[T; N, p]^+ = m_\mu \Phi^-_t(L_{N+1}, L_{N+2}, \dots, L_{N+p}).$$

□

Lemma 7.25. *For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $(i,k) \in \Gamma'(\mathbf{m})$, put $N = N^\mu_{(i,k)}$. Then we have the followings.*

(i) *If* $\mu_i^{(k)} \neq 0$, *we have*

$$\begin{aligned}
&m_\mu L_N^t [T; N-1, \mu_{i+1}^{(k)}+1]^+ [T; N, \mu_i^{(k)}]^- \\
&= q^{2\mu_i^{(k)}-2} m_\mu \Phi_t^+ (L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\quad + \delta_{(\mu_{i+1}^{(k)} \neq 0)} m_\mu L_N^t \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) [T; N, \mu_{i+1}^{(k)}]^+
\end{aligned}$$

(ii) *If* $\mu_i^{(k)} \neq 0$, *we have*

$$\begin{aligned}
&m_\mu L_N^t [T; N-1, \mu_{i+1}^{(k)}+1]^+ L_N [T; N, \mu_i^{(k)}]^- \\
&= q^{2\mu_i^{(k)}-2} m_\mu \Phi_{t+1}^+ (L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\quad - \delta_{(\mu_{i+1}^{(k)} \neq 0)} (q-q^{-1}) q^{2\mu_i^{(k)}-1} m_\mu \Phi_t^+ (L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\qquad \times \Phi_1^- (L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad + m_\mu L_N^t L_{N+1} \big([T; N-1, \mu_{i+1}^{(k)}+1]^+ - 1 \big) [T; N, \mu_i^{(k)}]^-
\end{aligned}$$

(iii) *If* $\mu_{i+1}^{(k)} \neq 0$, *we have*

$$\begin{aligned}
&m_\mu [T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ \\
&= (1 + \delta_{(t\neq 0)} (q^{2\mu_i^{(k)}} - 1)) m_\mu \Phi_t^- (L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad + \delta_{(\mu_i^{(k)} \neq 0)} (q-q^{-1}) \sum_{b=1}^{t-1} m_\mu q^{2\mu_i^{(k)}-1} \Phi_{t-b}^+ (L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\qquad \times \Phi_b^- (L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad + m_\mu L_N^t \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) [T; N, \mu_{i+1}^{(k)}]^+
\end{aligned}$$

(iv) *If* $\mu_{i+1}^{(k)} \neq 0$, *we have*

$$\begin{aligned}
&m_\mu L_{N+1} [T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ \\
&= (1 + \delta_{(t\neq 0)} (q^{2\mu_i^{(k)}} - 1)) m_\mu \Phi_{t+1}^- (L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad + \delta_{(\mu_i^{(k)} \neq 0)} (q-q^{-1}) \sum_{b=1}^{t-1} m_\mu q^{2\mu_i^{(k)}-1} \Phi_{t-b}^+ (L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\qquad \times \Phi_{b+1}^- (L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad + m_\mu L_N^t L_{N+1} \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) [T; N, \mu_{i+1}^{(k)}]^+
\end{aligned}$$

Proof. By the induction on $\mu_{i+1}^{(k)}$, we can prove that

$$\begin{aligned}
&[T; N-1, \mu_{i+1}^{(k)}+1]^+ [T; N, \mu_i^{(k)}]^- \\
&= [T; N, \mu_i^{(k)}]^- + \delta_{(\mu_{i+1}^{(k)} \neq 0)} \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) [T; N, \mu_{i+1}^{(k)}]^+
\end{aligned} \tag{7.25.1}$$

Thus we have

$$\begin{aligned}
&m_\mu L_N^t [T; N-1, \mu_{i+1}^{(k)}+1]^+ [T; N, \mu_i^{(k)}]^- \\
&= m_\mu L_N^t \Big\{ [T; N, \mu_i^{(k)}]^- + \delta_{(\mu_{i+1}^{(k)} \neq 0)} \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) [T; N, \mu_{i+1}^{(k)}]^+ \Big\}.
\end{aligned}$$

Applying Lemma 7.24 (i), we have (i).

We prove (ii). By Lemma 6.3, we have

$$\begin{aligned}
&[T; N-1, \mu_{i+1}^{(k)}+1]^+ L_N \\
&= L_N + L_{N+1} \big([T; N-1, \mu_{i+1}^{(k)}+1]^+ - 1 \big) - \delta_{(\mu_{i+1}^{(k)} \neq 0)} q(q-q^{-1}) L_{N+1} [T; N, \mu_{i+1}^{(k)}]^+.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&m_\mu L_N^t [T; N-1, \mu_{i+1}^{(k)}+1]^+ L_N [T; N, \mu_i^{(k)}]^- \\
&= m_\mu L_N^{t+1} [T; N, \mu_i^{(k)}]^- + m_\mu L_N^t L_{N+1} \big([T; N-1, \mu_{i+1}^{(k)}+1]^+ - 1 \big) [T; N, \mu_i^{(k)}]^- \\
&\quad - \delta_{(\mu_{i+1}^{(k)} \neq 0)} q(q-q^{-1}) m_\mu L_N^t L_{N+1} [T; N, \mu_{i+1}^{(k)}]^+ [T; N, \mu_i^{(k)}]^-.
\end{aligned}$$

Applying (6.4.2), Lemma 7.11, Lemma 7.24 and (7.25.1), we have (ii).

We prove (iii). By Lemma 6.3, we have

$$\begin{aligned}
[T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t &= L_{N+1}^t + L_N^t \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) \\
&\quad + \delta_{(\mu_i^{(k)} \neq 0)} q(q-q^{-1}) \sum_{b=1}^{t} L_N^{t-b} L_{N+1}^b [T; N, \mu_i^{(k)}]^-.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&m_\mu [T; N+1, \mu_i^{(k)}+1]^- L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ \\
&= m_\mu L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ + m_\mu L_N^t \big([T; N+1, \mu_i^{(k)}+1]^- - 1 \big) [T; N, \mu_{i+1}^{(k)}]^+ \\
&\quad + \delta_{(\mu_i^{(k)} \neq 0)} q(q-q^{-1}) \sum_{b=1}^{t} m_\mu L_N^{t-b} L_{N+1}^b [T; N, \mu_i^{(k)}]^- [T; N, \mu_{i+1}^{(k)}]^+ \\
&= m_\mu L_{N+1}^t [T; N, \mu_{i+1}^{(k)}]^+ \\
&\quad + \delta_{(\mu_i^{(k)} \neq 0)} \delta_{(t \neq 0)} q(q-q^{-1}) m_\mu L_{N+1}^t [T; N, \mu_i^{(k)}]^- [T; N, \mu_{i+1}^{(k)}]^+
\end{aligned}$$

$$+\,\delta_{(\mu_i^{(k)}\neq 0)} q(q-q^{-1})\sum_{b=1}^{t-1} m_\mu L_N^{t-b} L_{N+1}^{b}[T;N,\mu_i^{(k)}]^-[T;N,\mu_{i+1}^{(k)}]^+$$
$$+\,m_\mu L_N^t\big([T;N+1,\mu_i^{(k)}+1]^- -1\big)[T;N,\mu_{i+1}^{(k)}]^+$$

Applying (6.4.2), Lemma 7.11 and Lemma 7.24, we have (iii).

We prove (iv). By Lemma 6.3, we have

$$\begin{aligned}
&m_\mu L_{N+1}[T;N+1,\mu_i^{(k)}+1]^- L_{N+1}^t[T;N,\mu_{i+1}^{(k)}]^+\\
&= m_\mu L_{N+1}^{t+1}[T;N,\mu_{i+1}^{(k)}]^+ + m_\mu L_N^t L_{N+1}\big([T;N+1,\mu_i^{(k)}+1]^- -1\big)[T;N,\mu_{i+1}^{(k)}]^+\\
&\quad + \delta_{(\mu_i^{(k)}\neq 0)} q(q-q^{-1})\sum_{b=1}^{t} m_\mu L_N^{t-b}L_{N+1}^{b+1}[T;N,\mu_i^{(k)}]^-[T;N,\mu_{i+1}^{(k)}]^+
\end{aligned}$$

Applying (6.4.2), Lemma 7.11 and Lemma 7.24, we have (iv). □

Proposition 7.26. *For $(i,k)\in \varGamma'(\mathbf{m})$ and $s,t\geq 0$, we have*

$$[\mathcal{X}^+_{(i,k),t},\mathcal{X}^-_{(i,k),s}] = \begin{cases} \widetilde{\mathcal{K}}^+_{(i,k)}\mathcal{J}_{(i,k),s+t} & \text{if } i\neq m_k,\\ -Q_k\widetilde{\mathcal{K}}^+_{(m_k,k)}\mathcal{J}_{(m_k,k),s+t}+\widetilde{\mathcal{K}}^+_{(m_k,k)}\mathcal{J}_{(m_k,k),s+t+1} & \text{if } i=m_k.\end{cases}$$

Proof. Assume that $s=0$ and $t\geq 0$. For $\mu\in\varLambda_{n,r}(\mathbf{m})$, put $N=N^\mu_{(i,k)}$. By Lemma 7.17, we have

$$\begin{aligned}
&\mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(i,k),0}(m_\mu)\\
&= \delta_{(\mu_i^{(k)}\neq 0)} q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1} m_\mu L_N^t[T;N-1,\mu_{i+1}^{(k)}+1]^+ h^\mu_{-(i,k)}[T;N,\mu_i^{(k)}]^-
\end{aligned}\tag{7.26.1}$$

and

$$\begin{aligned}
&\mathcal{X}^-_{(i,k),0}\mathcal{X}^+_{(i,k),t}(m_\mu)\\
&= \delta_{(\mu_{i+1}^{(k)}\neq 0)} q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1} m_\mu h^{\mu+\alpha_{(i,k)}}_{-(i,k)}[T;N+1,\mu_i^{(k)}+1]^- L_{N+1}^t[T;N,\mu_{i+1}^{(k)}]^+.
\end{aligned}\tag{7.26.2}$$

Assume that $i\neq m_k$. By (7.26.1) and (7.26.2) together with Lemma 7.25, we have

$$\begin{aligned}
&(\mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(i,k),0}-\mathcal{X}^-_{(i,k),0}\mathcal{X}^+_{(i,k),t})(m_\mu)\\
&= q^{-\mu_i^{(k)}-\mu_{i+1}^{(k)}+1} m_\mu\Big\{\delta_{(\mu_i^{(k)}\neq 0)} q^{2\mu_i^{(k)}-2}\Phi_t^+(L_N,L_{N-1},\dots,L_{N-\mu_i^{(k)}+1})\\
&\quad - \delta_{(\mu_{i+1}^{(k)}\neq 0)}(1+\delta_{(t\neq 0)}(q^{2\mu_i^{(k)}}-1))\Phi_t^-(L_{N+1},L_{N+2},\dots,L_{N+\mu_{i+1}^{(k)}})\\
&\quad - \delta_{(\mu_i^{(k)}\neq 0)}\delta_{(\mu_{i+1}^{(k)}\neq 0)}(q-q^{-1})\sum_{b=1}^{t-1} q^{2\mu_i^{(k)}-1}\Phi_{t-b}^+(L_N,L_{N-1},\dots,L_{N-\mu_i^{(k)}+1})
\end{aligned}$$

$$
\begin{aligned}
&\qquad \times \Phi_b^-(L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}})\Big\} \\
&= q^{\mu_i^{(k)}-\mu_{i+1}^{(k)}} m_\mu \Big\{ \delta_{(\mu_i^{(k)}\neq 0)} q^{-t} q^{t-1} \Phi_t^+(L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\quad - \delta_{(\mu_{i+1}^{(k)}\neq 0)} (q^{-2\mu_i^{(k)}} + \delta_{(t\neq 0)}(1-q^{-2\mu_i^{(k)}})) q^t q^{-t+1} \Phi_t^-(L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}}) \\
&\quad - \delta_{(\mu_i^{(k)}\neq 0)} \delta_{(\mu_{i+1}^{(k)}\neq 0)} (q-q^{-1}) \sum_{b=1}^{t-1} q^{-t+2b} q^{t-b-1} \Phi_{t-b}^+(L_N, L_{N-1}, \dots, L_{N-\mu_i^{(k)}+1}) \\
&\qquad \times q^{-b+1} \Phi_b^-(L_{N+1}, L_{N+2}, \dots, L_{N+\mu_{i+1}^{(k)}})\Big\} \\
&= \widetilde{\mathcal{K}}_{(i,k)}^+ \mathcal{J}_{(i,k),t}(m_\mu).
\end{aligned}
$$

Thus, we have $[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(i,k),0}] = \widetilde{\mathcal{K}}^+_{(i,k)} \mathcal{J}_{(i,k),t}$ if $i \neq m_k$. (Note Corollary 7.9 in the case where $t = 0$.)

In a similar way, by (7.26.1) and (7.26.2) together with Lemma 7.25, we also have $[\mathcal{X}^+_{(m_k,k),t}, \mathcal{X}^-_{(m_k,k),0}] = -Q_k \widetilde{\mathcal{K}}^+_{(m_k,k)} \mathcal{J}_{(m_k,k),s+t} + \widetilde{\mathcal{K}}^+_{(m_k,k)} \mathcal{J}_{(m_k,k),s+t+1}$ if $i = m_k$. Now we proved the proposition in the case where $s = 0$ and $t \geq 0$.

Finally, we prove the proposition by the induction on s. In the case where $s = 0$, we have already proved. Assume that $s > 0$, by (7.15.1), we have

$$
\begin{aligned}
[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(i,k),s}] =& \mathcal{X}^+_{(i,k),t}(-\mathcal{I}^-_{(i,k),1}\mathcal{X}^-_{(i,k),s-1} + \mathcal{X}^-_{(i,k),s-1}\mathcal{I}^-_{(i,k),1}) \\
& - (-\mathcal{I}^-_{(i,k),1}\mathcal{X}^-_{(i,k),s-1} + \mathcal{X}^-_{(i,k),s-1}\mathcal{I}^-_{(i,k),1})\mathcal{X}^+_{(i,k),t}.
\end{aligned}
$$

Applying Proposition 7.21 together with Lemma 7.20, we have

$$
\begin{aligned}
[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(i,k),s}] &= -\mathcal{I}^-_{(i,k),1}\mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(i,k),s-1} + \mathcal{X}^+_{(i,k),t+1}\mathcal{X}^-_{(i,k),s-1} + \mathcal{X}^+_{(i,k),t}\mathcal{X}^-_{(i,k),s-1}\mathcal{I}^-_{(i,k),1} \\
&\quad + \mathcal{I}^-_{(i,k),1}\mathcal{X}^-_{(i,k),s-1}\mathcal{X}^+_{(i,k),t} - \mathcal{X}^-_{(i,k),s-1}\mathcal{X}^+_{(i,k),t}\mathcal{I}^-_{(i,k),1} - \mathcal{X}^-_{(i,k),s-1}\mathcal{X}^+_{(i,k),t+1} \\
&= [\mathcal{X}^+_{(i,k),t+1}, \mathcal{X}^-_{(i,k),s-1}] \\
&\quad - \mathcal{I}^-_{(i,k),1}[\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(i,k),s-1}] + [\mathcal{X}^+_{(i,k),t}, \mathcal{X}^-_{(i,k),s-1}]\mathcal{I}^-_{(i,k),1}.
\end{aligned}
$$

Then, by the assumption of the induction together with Lemma 7.6, we have the proposition. □

Lemma 7.27. *For $(i,k) \in \Gamma'(\mathbf{m})$, we have the followings.*

(i) *If $(q-q^{-1})$ is invertible in R, we have*

$$
\widetilde{\mathcal{K}}^+_{(i,k)} \mathcal{J}_{(i,k),0} = \frac{\widetilde{\mathcal{K}}^+_{(i,k)} - \widetilde{\mathcal{K}}^-_{(i,k)}}{q-q^{-1}}.
$$

(ii) *If $q = 1$, we have*

$$
\widetilde{\mathcal{K}}^+_{(i,k)} \mathcal{J}_{(i,k),0} = \mathcal{I}^+_{(i,k),0} - \mathcal{I}^-_{(i+1,k),0}.
$$

Proof. For $\mu \in \Lambda_{n,r}(\mathbf{m})$, by the definitions together with Corollary 7.9, we have

$$\begin{aligned}\widetilde{\mathcal{K}}^{+}_{(i,k)}\mathcal{J}_{(i,k),0}(m_\mu) &= \widetilde{\mathcal{K}}^{+}_{(i,k)}\big(\mathcal{I}^{+}_{(i,k),0} - (\mathcal{K}^{-}_{(i,k)})^2\mathcal{I}^{-}_{(i+1,k),0}\big)(m_\mu) \\ &= q^{\mu_i^{(k)}-\mu_{i+1}^{(k)}}(q^{-\mu_i^{(k)}}[\mu_i^{(k)}] - q^{-2\mu_i^{(k)}}q^{\mu_{i+1}^{(k)}}[\mu_{i+1}^{(k)}])m_\mu \\ &= [\mu_i^{(k)} - \mu_{i+1}^{(k)}]m_\mu.\end{aligned}$$

If $(q-q^{-1})$ is invertible in R, we have

$$\begin{aligned}[\mu_i^{(k)} - \mu_{i+1}^{(k)}]m_\mu &= \frac{q^{\mu_i^{(k)}-\mu_{i+1}^{(k)}} - q^{-\mu_i^{(k)}+\mu_{i+1}^{(k)}}}{q-q^{-1}}m_\mu \\ &= \frac{\widetilde{\mathcal{K}}^{+}_{(i,k)} - \widetilde{\mathcal{K}}^{-}_{(i,k)}}{q-q^{-1}}(m_\mu).\end{aligned}$$

Thus, we have (i).

If $q=1$, we have

$$\begin{aligned}[\mu_i^{(k)} - \mu_{i+1}^{(k)}]m_\mu &= (\mu_i^{(k)} - \mu_{i+1}^{(k)})m_\mu \\ &= (\mathcal{I}^{+}_{(i,k),0} - \mathcal{I}^{-}_{(i+1),0})(m_\mu).\end{aligned}$$

Thus, we have (ii). □

In the case where $q=1$, we have the following lemma.

Lemma 7.28. *Assume that $q=1$. Then, for $(j,l) \in \Gamma(\mathbf{m})$ and $t \geq 0$, we have the followings.*

(i) $\mathcal{K}^{\pm}_{(j,l)} = 1$.

(ii) $\mathcal{I}^{+}_{(j,l),t} = \mathcal{I}^{-}_{(j,l),t}$.

Proof. If $q=1$, we see that

$$\Phi_t^{\pm}(x_1,\dots,x_k) = x_1^t + x_2^t + \cdots + x_k^t, \tag{7.28.1}$$

in particular we have $\Phi_t^{+}(x_1,\dots,x_k) = \Phi_t^{-}(x_1,\dots,x_k)$. Thus, we have the lemma from the definitions. □

§ 8. The cyclotomic q-Schur algebra as a quotient of $\mathcal{U}_{q,\mathbf{Q}}(\mathbf{m})$

Let $\widetilde{\mathbf{Q}} = (Q_0, Q_1, \dots, Q_{r-1})$ be an r-tuple of indeterminate elements over $\mathbb{Z}$, and $\mathbb{Q}(\widetilde{\mathbf{Q}}) = \mathbb{Q}(Q_0, Q_1, \dots, Q_{r-1})$ be the quotient field of $\mathbb{Z}[\widetilde{\mathbf{Q}}] = \mathbb{Z}[Q_0, Q_1, \dots, Q_{r-1}]$. Put $\widetilde{\mathbb{A}} = \mathbb{Z}[q, q^{-1}, Q_0, Q_1, \dots, Q_{r-1}]$, and let $\widetilde{\mathbb{K}} = \mathbb{Q}(q, Q_0, Q_1, \dots, Q_{r-1})$ be the quotient field of $\widetilde{\mathbb{A}}$, where q is indeterminate over $\mathbb{Z}$. Put

$$\begin{aligned}&\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}) = \mathbb{Q}(\widetilde{\mathbf{Q}}) \otimes_{\mathbb{Q}(\mathbf{Q})} \mathfrak{g}_{\mathbf{Q}}(\mathbf{m}), \\ &\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{K}} \otimes_{\mathbb{K}} \mathcal{U}_{q,\mathbf{Q}}(\mathbf{m}) \text{ and } \mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) = \widetilde{\mathbb{A}} \otimes_{\mathbb{A}} \mathcal{U}_{\mathbb{A},q,\mathbf{Q}}(\mathbf{m}).\end{aligned}$$

We define a full subcategory $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ (resp. $\mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$) of $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-mod (resp. $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$-mod) in a similar manner as $\mathcal{C}_{\mathbf{Q}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{\mathbf{Q}}(\mathbf{m})$ (resp. $\mathcal{C}_{q,\mathbf{Q}}(\mathbf{m})$ and $\mathcal{C}^{\geq 0}_{q,\mathbf{Q}}(\mathbf{m})$).

Let $\mathscr{H}^{\widetilde{\mathbb{K}}}_{n,r}$ (resp. $\mathscr{H}^{\widetilde{\mathbb{A}}}_{n,r}$) be the Ariki-Koike algebra over $\widetilde{\mathbb{K}}$ (resp. over $\widetilde{\mathbb{A}}$) with parameters $q, Q_0, Q_1, \dots, Q_{r-1}$, and $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ (resp. $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$) be the cyclotomic q-Schur algebra associated with $\mathscr{H}^{\widetilde{\mathbb{K}}}_{n,r}$ (resp. $\mathscr{H}^{\widetilde{\mathbb{A}}}_{n,r}$). Then, we have the following theorem.

Theorem 8.1. *We have a homomorphism of algebras*

$$\Psi : \mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m}) \tag{8.1.1}$$

by taking $\Psi(\mathcal{X}^{\pm}_{(i,k),t}) = \mathcal{X}^{\pm}_{(i,k),t}$, $\Psi(\mathcal{I}^{\pm}_{(j,l),t}) = \mathcal{I}^{\pm}_{(j,l),t}$ *and* $\Psi(\mathcal{K}^{\pm}_{(j,l)}) = \mathcal{K}^{\pm}_{(j,l)}$.

The restriction of Ψ *to* $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ *gives a homomorphism of algebras*

$$\Psi_{\widetilde{\mathbb{A}}} : \mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m}) \to \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m}).$$

Moreover, if $m_k \geq n$ *for all* $k = 1, 2, \dots, r-1$*, the homomorphism* Ψ *(resp.* $\Psi_{\widetilde{\mathbb{A}}}$*) is surjective.*

Proof. The well-definedness of Ψ follows from Lemma 7.6, Lemma 7.7, Lemma 7.16, Proposition 7.18, Proposition 7.19, Lemma 7.20, Proposition 7.21, Proposition 7.22, and Proposition 7.26.

Note that $\mathscr{H}^{\mathbb{A}}_{n,r}$ (resp. $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$) is an $\mathbb{A}$-subalgebra of $\mathscr{H}^{\widetilde{\mathbb{K}}}_{n,r}$ (resp. $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$) by definitions. In particular, in order to see that $\varphi \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ belong to $\mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$, it is enough to show that $\varphi(m_\mu) \in \mathscr{H}^{\mathbb{A}}_{n,r}$ for any $\mu \in \Lambda_{n,r}(\mathbf{m})$.

For $\mu \in \Lambda_{n,r}(\mathbf{m})$ and $d \in \mathbb{Z}_{\geq 0}$, we see that,

$$\begin{bmatrix} \mathcal{K}_{(j,l)}; 0 \\ d \end{bmatrix} (m_\mu) = \begin{cases} \begin{bmatrix} \mu^{(l)}_j \\ d \end{bmatrix} m_\mu & \text{if } d \leq \mu^{(l)}_j, \\ 0 & \text{if } d > \mu^{(l)}_j \end{cases} \tag{8.1.2}$$

in $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$. This implies that $\Psi(\left[\begin{smallmatrix} \mathcal{K}_{(j,l)};0 \\ d \end{smallmatrix}\right]) \in \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$.

For $(i,k) \in \Gamma'(\mathbf{m})$ and $t, d \in \mathbb{Z}_{\geq 0}$, we see that

$$\begin{aligned}
&(\mathcal{X}^{+}_{(i,k),t})^d (m_\mu) \\
&= q^{-d\mu^{(k)}_{i+1} + d(d+1)/2} m_{\mu + d\alpha_{(i,k)}} (L_{N^{\mu}_{(i,k)}+1} L_{N^{\mu}_{(i,k)}+2} \dots L_{N^{\mu}_{(i,k)}+d})^t \left[\begin{smallmatrix} T; N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1} \\ d \end{smallmatrix}\right] \\
&= q^{-d\mu^{(k)}_{i+1} + d(d+1)/2} m_{\mu + d\alpha_{(i,k)}} (L_{N^{\mu}_{(i,k)}+1} L_{N^{\mu}_{(i,k)}+2} \dots L_{N^{\mu}_{(i,k)}+d})^t \\
&\quad \times (T; N^{\mu}_{(i,k)}, d)^{+}! \mathfrak{H}^{+}(N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}, d)
\end{aligned}$$

by Lemma 7.17 together with Lemma 7.11 and Corollary 7.14. We also see that $(T; N^{\mu}_{(i,k)}, d)^+!$ commute with $(L_{N^{\mu}_{(i,k)}+1} L_{N^{\mu}_{(i,k)}+2} \dots L_{N^{\mu}_{(i,k)}+d})^t$ by Lemma 6.3 (iii), and see that $m_{\mu+d\alpha_{(i,k)}}(T; N^{\mu}_{(i,k)}, d)^+! = q^{d(d-1)/2}[d]! m_{\mu+d\alpha_{(i,k)}}$ by (6.4.2). Thus we have

$$
\begin{aligned}
&(\mathcal{X}^+_{(i,k),t})^d (m_\mu) \\
&= [d]! q^{-d\mu^{(k)}_{i+1}+d^2} m_{\mu+d\alpha_{(i,k)}} (L_{N^{\mu}_{(i,k)}+1} L_{N^{\mu}_{(i,k)}+2} \dots L_{N^{\mu}_{(i,k)}+d})^t \mathfrak{H}^+(N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}, d)
\end{aligned}
$$

in $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$. This implies that $\Psi(\mathcal{X}^{+(d)}_{(i,k)t}) \in \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$ since $\mathfrak{H}^+(N^{\mu}_{(i,k)}, \mu^{(k)}_{i+1}, d) \in \mathscr{H}^{\mathbb{A}}_{n,r}$ by the argument in the proof of Corollary 7.14. Similarly, we see that $\Psi(\mathcal{X}^{-(d)}_{(i,k),t}) \in \mathscr{S}^{\widetilde{\mathbb{A}}}_{n,r}(\mathbf{m})$. Thus, the restriction of Ψ to $\mathcal{U}_{\widetilde{\mathbb{A}},q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ gives a homomorphism $\Psi_{\widetilde{\mathbb{A}}}$.

The last assertion follows from [W1, Proposition 6.4]. $\square$

Remark 8.2. In order to prove the surjectivity of Ψ (resp. $\Psi_{\widetilde{\mathbb{A}}}$), we use the result of [W1, Proposition 6.4]. In fact, we considered only the case where $m_k = n$ for all $k = 1, 2, \dots, r$ in [W1]. However, we can apply the result to the case where $m_k \geq n$ for all $k = 1, 2, \dots, r-1$ without any change since the surjectivity in [W1, Proposition 6.4] follows from the result in [DR]. The reason why we assume the condition $m_k \geq n$ for all $k = 1, 2, \dots, r-1$ to state the surjectivity of Ψ is just the using results of [DR]. We expect that Ψ is also surjective without this condition.

Theorem 8.3. *Assume that $m_k \geq n$ for all $k = 1, 2, \dots, r-1$. Then we have the followings.*

(i) $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$-mod *is a full subcategory of $\mathcal{C}^{\geq 0}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the surjection Ψ in* (8.1.1).

(ii) *The Weyl module $\Delta(\lambda) \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$*-mod *($\lambda \in \Lambda^+_{n,r}(\mathbf{m})$) is the simple highest weight $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$-module of highest weight $(\lambda, \boldsymbol{\varphi})$ through the surjection Ψ, where the multiset $\boldsymbol{\varphi} = (\varphi^{\pm}_{(j,l),t} \in \widetilde{\mathbb{K}} \mid (j,l) \in \Gamma(\mathbf{m}), t \geq 1)$ is given by*

$$
\varphi^+_{(j,l),t} = Q^t_{l-1} q^{(2t-1)\lambda^{(l)}_j - t(2j-1)} [\lambda^{(l)}_j] \ \textit{and}\ \varphi^-_{(j,l),t} = Q^t_{l-1} q^{\lambda^{(l)}_j - t(2j-1)} [\lambda^{(l)}_j].
$$

Proof. For $\lambda \in \Lambda_{n,r}(\mathbf{m})$, let 1_λ be an element of $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ such that the identity on M^λ and $1_\lambda(M^\mu) = 0$ for any $\mu \neq \lambda$. Then we have $1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda$ and $\sum_{\lambda \in \Lambda_{n,r}(\mathbf{m})} 1_\lambda = 1$. Thus, for $M \in \mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}$-mod, we have the decomposition

$$
M = \bigoplus_{\mu \in \Lambda_{n,r}(\mathbf{m})} 1_\mu M. \tag{8.3.1}
$$

Moreover, we see that

$$
1_\mu M = \{ m \in M \mid \mathcal{K}^+_{(j,l)} \cdot m = q^{\mu^{(l)}_j} m \text{ for } (j,l) \in \Gamma(\mathbf{m}) \}
$$

from the definition of Ψ. Thus, any object M of $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}$-mod has the weight space decomposition (8.3.1) as a $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$-module, where we remark that $\Lambda_{n,r}(\mathbf{m}) \subset P_{\geq 0}$.

For $M \in \mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m})$-mod, in order to see that all eigenvalues of the action of $\mathcal{I}_{(j,l),t}^{\pm}$ $((j,l) \in \Gamma(\mathbf{m}), t \geq 0)$ on M belong to $\widetilde{\mathbb{K}}$, it is enough to show them for $\Delta(\lambda)$ $(\lambda \in \Lambda_{n,r}^{+}(\mathbf{m}))$ since $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}(\mathbf{m})$ is semi-simple and $\{\Delta(\lambda) \mid \lambda \in \Lambda_{n,r}^{+}(\mathbf{m})\}$ gives a complete set of isomorphism classes of simple $\mathscr{S}_{n,r}^{\widetilde{\mathbb{K}}}$-modules. Recall that $\{\varphi_T \mid T \in \mathcal{T}_0(\lambda,\mu)$ for some $\mu \in \Lambda_{n,r}(\mathbf{m})\}$ gives a basis of $\Delta(\lambda)$.

Note that $\Phi_t^{\pm}(L_{N_{(j,l)}^{\mu}}, L_{N_{(j,l)}^{\mu}-1}, \dots, L_{N_{(j,l)}^{\mu}-\mu_j^{(l)}+1})$ commute with T_w for any $w \in \mathfrak{S}_{\mu}$ by Lemma 6.3, for $T \in \mathcal{T}_0(\lambda,\mu)$, we have

$$(8.3.2)\quad \mathcal{I}_{(j,l),t}^{\pm} \cdot \varphi_T = \begin{cases} q^{\pm(t-1)}\Phi_t^{\pm}(\mathrm{res}_{(j,l);T})\varphi_T + \sum\limits_{S \rhd T} r_S \varphi_S \quad (r_S \in \widetilde{\mathbb{K}}) & \text{if } \mu_j^{(l)} \neq 0, \\ 0 & \text{if } \mu_j^{(l)} = 0 \end{cases}$$

in a similar argument as in the proof of [JM, Theorem 3.10], where

$$\Phi_t^{\pm}(\mathrm{res}_{(j,l);T}) = \Phi_t^{\pm}(\mathrm{res}(x_1), \mathrm{res}(x_2), \dots, \mathrm{res}(x_{\mu_j^{(l)}}))$$

with $\{x_1, x_2, \dots, x_{\mu_j^{(l)}}\} = \{x \in [\lambda] \mid T(x) = (j,l)\}$, and $\rhd$ is a partial order on $\mathcal{T}_0(\lambda,\mu)$ defined in [JM, Definition 3.6]. This implies that all eigenvalues of the action of $\mathcal{I}_{(j,l),t}^{\pm}$ on $\Delta(\lambda)$ belong to $\widetilde{\mathbb{K}}$. Now we proved (i).

We prove (ii). For $\lambda \in \Lambda_{n,r}^{+}(\mathbf{m})$, let T^{λ} be the unique semi-standard tableau of shape λ with weight λ. Then, we see easily that $\varphi_{T^{\lambda}}$ is a highest weight vector of $\Delta(\lambda)$. Note that there is no tableau such that $S \rhd T^{\lambda}$, then we have

$$(8.3.3)\qquad \varphi_{(j,l),t}^{\pm} = q^{\pm(t-1)}\Phi_t^{\pm}(Q_k q^{2(1-j)}, Q_k q^{2(2-j)}, \dots, Q_k q^{2(\lambda_j^{(l)}-j)})$$

by (8.3.2). Then we can prove (ii) by the induction on t using (8.3.3) and (7.3.1). □

Let $\mathscr{S}_{n,r}^{\mathbf{1}}(\mathbf{m})$ be the cyclotomic q-Schur algebra over $\mathbb{Q}(\widetilde{\mathbf{Q}})$ with parameters $q = 1, Q_0, Q_1, \dots, Q_{r-1}$. Then we have the following theorem.

Theorem 8.4.

(i) *We have a homomorphism of algebras*

$$(8.4.1)\qquad \Psi_{\mathbf{1}} : U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m})) \to \mathscr{S}_{n,r}^{\mathbf{1}}(\mathbf{m})$$

by taking $\Psi_{\mathbf{1}}(\mathcal{X}_{(i,k),t}^{\pm}) = \mathcal{X}_{(i,k),t}^{\pm}$ *and* $\Psi_{\mathbf{1}}(\mathcal{I}_{(j,l),t}) = \mathcal{I}_{(j,l),t}^{+}(= \mathcal{I}_{(j,l),t}^{-})$.

Moreover, if $m_k \geq n$ *for all* $k = 1, 2, \dots, r-1$, *the homomorphism* $\Psi_{\mathbf{1}}$ *is surjective.*

(ii) *Assume that* $m_k \geq n$ *for all* $k = 1, 2, \dots, r-1$. *Then* $\mathscr{S}_{n,r}^{\mathbf{1}}(\mathbf{m})$-mod *is a full subcategory of* $\mathcal{C}_{\widetilde{\mathbf{Q}}}^{\geq 0}(\mathbf{m})$ *through the surjection* $\Psi_{\mathbf{1}}$.

Moreover, the Weyl module $\Delta(\lambda) \in \mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod *(*$\lambda \in \Lambda^+_{n,r}$*) is the simple highest weight* $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$*-module of highest weight* $(\lambda, \boldsymbol{\varphi})$ *through the surjection* $\Psi_{\mathbf{1}}$*, where the multiset* $\boldsymbol{\varphi} = (\varphi_{(j,l),t} \in \mathbb{Q}(\widetilde{\mathbf{Q}}) \,|\, (j,l) \in \Gamma(\mathbf{m}), t \geq 1)$ *is given by*

$$\varphi_{(j,l),t} = Q^t_{l-1} \lambda^{(l)}_j .$$

Proof. Note Lemma 7.27 and Lemma 7.28, then we can prove the theorem in a similar way as in the proof of Theorem 8.1 and Theorem 8.3. □

§ 9. Characters of Weyl modules of cyclotomic q-Schur algebras

In this section, we study the characters of Weyl modules of cyclotomic q-Schur algebras as symmetric polynomials. In particular, we prove the conjecture given in [W2] (the formula (9.2.1) below) which will be understood as the decomposition of the tensor product of Weyl modules in the case where $q = 1$.

9.1. Characters. For $k = 1, \dots, r$, let $\mathbf{x}^{(k)}_{\mathbf{m}} = (x_{(1,k)}, x_{(2,k)}, \dots, x_{(m_k,k)})$ be the set of m_k independent variables, and put $\mathbf{x}_{\mathbf{m}} = \cup_{k=1}^r \mathbf{x}^{(k)}_{\mathbf{m}}$. Let $\mathbb{Z}[\mathbf{x}^{\pm}_{\mathbf{m}}]$ (resp. $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$) be the ring of Laurent polynomials (resp. the ring of polynomials) with variables $\mathbf{x}_{\mathbf{m}}$. For $\lambda \in P$, we define the monomial $x^\lambda \in \mathbb{Z}[\mathbf{x}^{\pm}_{\mathbf{m}}]$ by $x^\lambda = \prod_{k=1}^r \prod_{i=1}^{m_k} x_{(i,k)}^{\langle \lambda, h_{(i,k)} \rangle}$.

For $M \in \mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ (resp. $M \in \mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$), we define the character of M by

$$\operatorname{ch} M = \sum_{\lambda \in P} \dim M_\lambda x^\lambda \in \mathbb{Z}[\mathbf{x}^{\pm}_{\mathbf{m}}]. \tag{9.1.1}$$

It is clear that $\operatorname{ch} M \in \mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ if $M \in \mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ (resp. $M \in \mathcal{C}^{\geq 0}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$).

When we regard $M \in \mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$ as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module through the injection (2.16.2), $\operatorname{ch} M$ defined by (9.1.1) coincides with the character of M as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module since M_λ is also the weight space of weight λ as a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-module . Thus, by the known results for $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-modules, we see that

$$\operatorname{ch} M \in \bigotimes_{k=1}^r \mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}]^{\mathfrak{S}_{m_k}} \text{ if } M \in \mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m}),$$

where $\mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}]^{\mathfrak{S}_{m_k}}$ is the ring of symmetric polynomials with variables $\mathbf{x}^{(k)}_{\mathbf{m}}$, and we regard $\bigotimes_{k=1}^r \mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}]^{\mathfrak{S}_{m_k}}$ as a subring of $\mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ through the multiplication map $\bigotimes_{k=1}^r \mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}]^{\mathfrak{S}_{m_k}} \to \mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ $(\otimes_{k=1}^r f(\mathbf{x}^{(k)}_{\mathbf{m}}) \mapsto \prod_{k=1}^r f(\mathbf{x}^{(k)}_{\mathbf{m}}))$. It is similar for $M \in \mathcal{C}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ through the injection (4.9.2).

9.2. The character of the Weyl module $\Delta(\lambda) \in \mathscr{S}_{n,r}(\mathbf{m})$ ($\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$) is studied in [W2]. Note that $\operatorname{ch} \Delta(\lambda)$ ($\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$) does not depend on the choice of the base field and parameters. Put $\widetilde{\Lambda}^+_{\geq 0,r}(\mathbf{m}) = \cup_{n \geq 0} \widetilde{\Lambda}^+_{n,r}(\mathbf{m})$. For $\lambda, \mu \in \widetilde{\Lambda}^+_{\geq 0,r}(\mathbf{m})$, the

following formula was conjectured in [W2, Conjecture 2]:

$$\operatorname{ch}\Delta(\lambda)\operatorname{ch}\Delta(\mu) = \sum_{\nu\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})} \mathrm{LR}^{\nu}_{\lambda\mu}\operatorname{ch}\Delta(\nu) \ \text{ for } \lambda,\mu\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m}), \tag{9.2.1}$$

where $\mathrm{LR}^{\nu}_{\lambda\mu} = \prod_{k=1}^{r}\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}$, and $\mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}$ is the Littlewood-Richardson coefficient for the partitions $\lambda^{(k)}$, $\mu^{(k)}$ and $\nu^{(k)}$. We prove this conjecture as follows.

9.3. For $\lambda=(\lambda^{(1)},\dots,\lambda^{(r)})\in\widetilde{\varLambda}^+_{n,r}(\mathbf{m})$, we denote

$$(\underbrace{0,\dots,0}_{k-1},\lambda^{(k)},0,\dots,0)\in\widetilde{\varLambda}^+_{n_k,r}(\mathbf{m})$$

by $(0,\dots,\lambda^{(k)},\dots,0)$ simply, where $n_k=\sum_{i=1}^{m_k}\lambda^{(k)}_i$ (i.e. $\lambda^{(k)}$ appears in the k-th component in $(0,\dots,\lambda^{(k)},\dots,0)$). Let

$$S_{\lambda^{(k)}}(\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}})\in\mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}}]^{\mathfrak{S}(\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}})}$$

be the Schur polynomial for the partition $\lambda^{(k)}$ with variables $\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}}$, where we regard $\mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}}]^{\mathfrak{S}(\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}})}$ as a subring of $\bigotimes_{k=1}^{r}\mathbb{Z}[\mathbf{x}^{(k)}_{\mathbf{m}}]^{\mathfrak{S}_{m_k}}\subset\mathbb{Z}[\mathbf{x}_{\mathbf{m}}]$ in the natural way. Put $\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}})=\operatorname{ch}\Delta(\lambda)$ ($\lambda\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})$). Then we have the following proposition.

Proposition 9.4. *For $\lambda,\mu\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})$, we have the following formulas.*

(i) $\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}}) = S_{\lambda^{(k)}}(\mathbf{x}^{(k)}_{\mathbf{m}}\cup\dots\cup\mathbf{x}^{(r)}_{\mathbf{m}})$.

(ii) $\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}}) = \prod_{k=1}^{r}\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}})$.

(iii) $\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}})\widetilde{S}_{\mu}(\mathbf{x}_{\mathbf{m}}) = \sum_{\nu\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})}\mathrm{LR}^{\nu}_{\lambda\mu}\,\widetilde{S}_{\nu}(\mathbf{x}_{\mathbf{m}})$.

Proof. (i). By the definition of the cellular basis of $\mathscr{S}_{n,r}(\mathbf{m})$ in [DJM], for $\lambda\in\widetilde{\varLambda}^+_{n,r}(\mathbf{m})$, we have

$$\widetilde{S}_{\lambda}(\mathbf{x}_{\mathbf{m}}) = \operatorname{ch}\Delta(\lambda) = \sum_{\mu\in\varLambda_{n,r}(\mathbf{m})}\sharp\mathcal{T}_0(\lambda,\mu)x^{\mu}. \tag{9.4.1}$$

Thus, we have

$$\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x}_{\mathbf{m}}) = \sum_{\mu\in\varLambda_{n_k,r}(\mathbf{m})}\sharp\mathcal{T}_0((0,\dots,\lambda^{(k)},\dots,0),\mu)x^{\mu}, \tag{9.4.2}$$

where $n_k=\sum_{i=1}^{m_k}\lambda^{(k)}_i$. We see that

$$\mu^{(1)}=\dots=\mu^{(k-1)}=0 \text{ if } \mathcal{T}_0((0,\dots,\lambda^{(k)},\dots,0),\mu)\neq\emptyset$$

by the definition of semi-standard tableaux. Thus, we have $\widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x_m}) \in \bigotimes_{l=k}^{r} \mathbb{Z}[\mathbf{x}^{(l)}_{\mathbf{m}}]^{\mathfrak{S}_{m_k}}$. Put

$$\Lambda^{\geq k}_{n_k,r}(\mathbf{m}) = \{\mu = (\mu^{(1)},\dots,\mu^{(r)}) \in \Lambda_{n_k,r}(\mathbf{m}) \mid \mu^{(l)} = 0 \text{ for } l = 1,\dots,k-1\}.$$

Put $m' = m_k + \dots + m_r$. We identify the set $\Lambda_{n_k,1}(m')$ with $\Lambda^{\geq k}_{n_k,r}(\mathbf{m})$ by the bijection $\theta^k : \Lambda_{n_k,1}(m') \mapsto \Lambda^{\geq k}_{n_k,r}(\mathbf{m})$ such that

$$(\theta^k(\mu))^{(k+l)}_i = \begin{cases} \mu_i & \text{if } l = 0, \\ \mu_{m_k+m_{k+1}+\dots+m_{k+l-1}+i} & \text{if } 1 \leq l \leq r-k \end{cases}$$

for $\mu = (\mu_1,\mu_2,\dots,\mu_{m'}) \in \Lambda_{n_k,1}(m')$. By the well-known fact, we can describe the Schur polynomial $S_{\lambda^{(k)}}(\mathbf{x}^{(k)}_{\mathbf{m}} \cup \dots \cup \mathbf{x}^{(r)}_{\mathbf{m}})$ as

$$S_{\lambda^{(k)}}(\mathbf{x}^{(k)}_{\mathbf{m}} \cup \dots \cup \mathbf{x}^{(r)}_{\mathbf{m}}) = \sum_{\mu \in \Lambda_{n_k,1}(m')} \sharp\mathcal{T}_0(\lambda^{(k)},\mu)x^{\mu}, \tag{9.4.3}$$

where we put $x^{\mu} = \prod_{i=1}^{m_k} x_{(i,k)}^{\mu_i} \prod_{l=1}^{r-k}\prod_{i=1}^{m_l} x_{(i,k+l)}^{\mu_{m_k+m_{k+1}+\dots+m_{k+l-1}+i}}$. From the definition of semi-standard tableaux, we see that

$$\sharp\mathcal{T}_0(\lambda^{(k)},\mu) = \sharp\mathcal{T}_0((0,\dots,\lambda^{(k)},\dots,0),\theta^k(\mu))$$

for $\mu \in \Lambda_{n_k,1}(m')$. Thus, by comparing the right hand sides of (9.4.2) and of (9.4.3), we obtain (i).

(ii). First we prove that

$$\widetilde{S}_{(\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)})}(\mathbf{x_m}) = \widetilde{S}_{(\lambda^{(1)},0,\dots,0)}(\mathbf{x_m})\widetilde{S}_{(0,\lambda^{(2)},\dots,\lambda^{(r)})}(\mathbf{x_m}). \tag{9.4.4}$$

By (9.4.1), we have

$$\widetilde{S}_{(\lambda^{(1)},\lambda^{(2)},\dots,\lambda^{(r)})}(\mathbf{x_m}) = \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \sharp\mathcal{T}_0(\lambda,\mu)\, x^{\mu}. \tag{9.4.5}$$

On the other hand, we have

(9.4.6)
$$\begin{aligned}
&\widetilde{S}_{(\lambda^{(1)},0,\dots,0)}(\mathbf{x_m})\widetilde{S}_{(0,\lambda^{(2)},\dots,\lambda^{(r)})}(\mathbf{x_m}) \\
&= \Big(\sum_{\nu \in \Lambda_{n_1,r}(\mathbf{m})} \sharp\mathcal{T}_0((\lambda^{(1)},0,\dots,0),\nu)\, x^{\nu}\Big)\Big(\sum_{\tau \in \Lambda_{n',r}(\mathbf{m})} \sharp\mathcal{T}_0((0,\lambda^{(2)},\dots,\lambda^{(r)}),\tau)\, x^{\tau}\Big) \\
&= \sum_{\mu \in \Lambda_{n,r}(\mathbf{m})} \Big(\sum_{\substack{\nu \in \Lambda_{n_1,r}(\mathbf{m}), \tau \in \Lambda_{n',r}(\mathbf{m}) \\ \nu+\tau=\mu}} \sharp\mathcal{T}_0((\lambda^{(1)},0,\dots,0),\nu)\sharp\mathcal{T}_0((0,\lambda^{(2)},\dots,\lambda^{(r)}),\tau)\Big)x^{\mu}
\end{aligned}$$

where $n_1 = \sum_{i=1}^{m_1} \lambda_i^{(1)}$ and $n' = n - n_1$. From the definition of semi-standard tableaux, we can check that

$$(9.4.7) \qquad \sharp\mathcal{T}_0(\lambda,\mu) = \sum_{\substack{\nu\in\varLambda_{n_1,r}(\mathbf{m}),\tau\in\varLambda_{n',r}(\mathbf{m})\\ \nu+\tau=\mu}} \sharp\mathcal{T}_0((\lambda^{(1)},0,\dots,0),\nu)\sharp\mathcal{T}_0((0,\lambda^{(2)},\dots,\lambda^{(r)}),\tau).$$

Thus, (9.4.5), (9.4.6) and (9.4.7) imply (9.4.4). By applying a similar argument to $\widetilde{S}_{(0,\lambda^{(2)},\dots,\lambda^{(r)})}(\mathbf{x_m})$ inductively, we obtain (ii).

By (i) and (ii), we have

$$\begin{aligned}
\widetilde{S}_\lambda(\mathbf{x_m})\widetilde{S}_\mu(\mathbf{x_m}) &= \Big(\prod_{k=1}^r \widetilde{S}_{(0,\dots,\lambda^{(k)},\dots,0)}(\mathbf{x_m})\Big)\Big(\prod_{k=1}^r \widetilde{S}_{(0,\dots,\mu^{(k)},\dots,0)}(\mathbf{x_m})\Big)\\
&= \Big(\prod_{k=1}^r S_{\lambda^{(k)}}(\mathbf{x}_\mathbf{m}^{(k)}\cup\cdots\cup\mathbf{x}_\mathbf{m}^{(r)})\Big)\Big(\prod_{k=1}^r S_{\mu^{(k)}}(\mathbf{x}_\mathbf{m}^{(k)}\cup\cdots\cup\mathbf{x}_\mathbf{m}^{(r)})\Big)\\
&= \prod_{k=1}^r S_{\lambda^{(k)}}(\mathbf{x}_\mathbf{m}^{(k)}\cup\cdots\cup\mathbf{x}_\mathbf{m}^{(r)})\big)S_{\mu^{(k)}}(\mathbf{x}_\mathbf{m}^{(k)}\cup\cdots\cup\mathbf{x}_\mathbf{m}^{(r)})\big)\\
&= \prod_{k=1}^r \Big(\sum_{\nu^{(k)}\in\varLambda^+_{\geq 0,1}(m_k+\cdots+m_r)} \mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}\, S_{\nu^{(k)}}(\mathbf{x}_\mathbf{m}^{(k)}\cup\cdots\cup\mathbf{x}_\mathbf{m}^{(r)})\Big)\\
&= \sum_{\nu\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})} \Big(\prod_{k=1}^r \mathrm{LR}^{\nu^{(k)}}_{\lambda^{(k)}\mu^{(k)}}\Big)\prod_{k=1}^r \widetilde{S}_{(0,\dots,\nu^{(k)},\dots,0)}(\mathbf{x_m})\\
&= \sum_{\nu\in\widetilde{\varLambda}^+_{\geq 0,r}(\mathbf{m})} \mathrm{LR}^\nu_{\lambda\mu}\,\widetilde{S}_\nu(\mathbf{x_m}),
\end{aligned}$$

where we note that, if $\ell(\lambda^{(k)}) > m_k+\cdots+m_r$ for some k, we have $S_{\lambda^{(k)}}(\mathbf{x}_\mathbf{m}^{(k)}\cup\cdots\cup\mathbf{x}_\mathbf{m}^{(r)}) = 0$ and $\mathcal{T}_0(\lambda,\mu)=\emptyset$ for any $\mu\in\varLambda_{n,r}(\mathbf{m})$. Now we obtained (iii). $\square$

§ 10. Tensor products for Weyl modules of cyclotomic q-Schur algebras at $q=1$

By using the comultiplication $\Delta : U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m})) \to U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))\otimes U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))$ ($\Delta(x) = x\otimes 1 + 1\otimes x$), we define the $U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))$-module $M\otimes N$ for $U(\mathfrak{g}_{\widetilde{Q}}(\mathbf{m}))$-module M and N. We regard $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules ($n\geq 0$) as a $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules through the homomorphism $\Psi_\mathbf{1}$ in (8.4.1). Note that $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ is semi-simple, and $\{\Delta(\lambda)\mid\lambda\in\varLambda^+_{n,r}(\mathbf{m})\}$ gives a complete set of isomorphism classes of simple $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-modules if $m_k\geq n$ for all $k=1,2,\dots,r-1$. Then, we have the following proposition.

Proposition 10.1. *Assume that $m_k\geq n$ for all $k=1,2,\dots,r-1$. Take $n_1,n_2\in\mathbb{Z}_{>0}$ such that $n=n_1+n_2$. For $\lambda\in\varLambda^+_{n_1,r}(\mathbf{m})$ (resp. $\mu\in\varLambda^+_{n_2,r}(\mathbf{m})$), let $\Delta(\lambda)$ (resp. $\Delta(\mu)$) be the Weyl module of $\mathscr{S}^{\mathbf{1}}_{n_1,r}(\mathbf{m})$ (resp. $\mathscr{S}^{\mathbf{1}}_{n_2,r}(\mathbf{m})$) corresponding λ (resp. μ).*

Then we have

$$\Delta(\lambda)\otimes\Delta(\mu)\cong\bigoplus_{\nu\in\Lambda^+_{n,r}(\mathbf{m})}\mathrm{LR}^\nu_{\lambda\mu}\,\Delta(\nu)\ \text{ as } U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))\text{-modules,}\tag{10.1.1}$$

where $\Delta(\nu)$ is the Weyl module of $\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$ corresponding ν, and $\mathrm{LR}^\nu_{\lambda\mu}\,\Delta(\nu)$ means the direct sum of $\mathrm{LR}^\nu_{\lambda\mu}$ copies of $\Delta(\nu)$. In particular, $\Delta(\lambda)\otimes\Delta(\mu)\in\mathscr{S}^{\mathbf{1}}_{n,r}(\mathbf{m})$-mod.

Proof. For $\tau\in P_{\geq 0}$, put

$$\pi_{\mathbf{m}}(\tau)=(|\tau^{(1)}|,|\tau^{(2)}|,\dots,|\tau^{(r)}|)\in\mathbb{Z}^r_{\geq 0},$$

where $|\tau^{(l)}|=\sum_{j=1}^{m_l}\langle\tau,h_{(j,l)}\rangle$ for $l=1,\dots,r$. We denote by $\geq$ the lexicographic order on $\mathbb{Z}^r_{\geq 0}$. Then we have the weight space decomposition

$$\Delta(\lambda)\otimes\Delta(\mu)=\bigoplus_{\substack{\tau\in\Lambda_{n,r}(\mathbf{m})\\ \pi_{\mathbf{m}}(\tau)\leq\pi_{\mathbf{m}}(\lambda+\mu)}}(\Delta(\lambda)\otimes\Delta(\mu))_\tau.\tag{10.1.2}$$

On the other hand, it is clear that $\Delta(\lambda)\otimes\Delta(\mu)\in\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$. Thus, we have

$$[\Delta(\lambda)\otimes\Delta(\mu)]=\sum_{\substack{\nu\in\Lambda^+_{n,r}(\mathbf{m})\\ \pi_{\mathbf{m}}(\nu)\leq\pi_{\mathbf{m}}(\lambda+\mu)}}\sum_{\boldsymbol{\varphi}}d_{\nu,\boldsymbol{\varphi}}[L(\nu,\boldsymbol{\varphi})]\ \text{ in } K_0(\mathcal{C}^{\geq 0}_{\widetilde{\mathbf{Q}}}(\mathbf{m})),\tag{10.1.3}$$

where $d_{\nu,\boldsymbol{\varphi}}$ is the composition multiplicity of the simple highest weight $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-module $L(\nu,\boldsymbol{\varphi})$ of highest weight $(\nu,\boldsymbol{\varphi})$ in $\Delta(\lambda)\otimes\Delta(\mu)$.

Note that $L_{i+1}T_i=T_iL_i$ and $L_iT_i=T_iL_{i+1}$ since $q=1$. Then, for $(j,l)\in\Gamma(\mathbf{m})$ and $t\geq 1$, we see that

$$\mathcal{I}_{(j,l),t}\cdot v=Q^t_{l-1}\nu^{(l)}_j v\ \text{ for any } v\in(\Delta(\lambda)\otimes\Delta(\mu))_\nu\tag{10.1.4}$$

if $\pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu)$ by the argument in the proof of [JM, Proposition 3.7 and Theorem 3.10]. This implies that

$$L(\nu,\boldsymbol{\varphi})\cong\Delta(\nu)\ \text{ if } d_{\nu,\boldsymbol{\varphi}}\neq 0 \text{ and } \pi_{\mathbf{m}}(\nu)=\pi_{\mathbf{m}}(\lambda+\mu)\tag{10.1.5}$$

by Theorem 8.4 (ii). By Proposition 9.4 (iii) together with (10.1.3) and (10.1.5), we have

(10.1.6)

$$\begin{aligned}\mathrm{ch}(\Delta(\lambda)\otimes\Delta(\mu))&=\widetilde{S}_\lambda(\mathbf{x}_{\mathbf{m}})\widetilde{S}_\mu(\mathbf{x}_{\mathbf{m}})\\&=\sum_{\nu\in\Lambda^+_{n,r}(\mathbf{m})}\mathrm{LR}^\nu_{\lambda\mu}\,\widetilde{S}_\nu(\mathbf{x}_{\mathbf{m}})\end{aligned}$$

$$= \sum_{\substack{\nu \in \varLambda^+_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)}} d_\nu \widetilde{S}_\nu(\mathbf{x_m}) + \sum_{\substack{\nu \in \varLambda^+_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) < \pi_{\mathbf{m}}(\lambda+\mu)}} \sum_{\boldsymbol{\varphi}} d_{\nu,\boldsymbol{\varphi}} \operatorname{ch} L(\nu, \boldsymbol{\varphi}),$$

where d_ν is the composition multiplicity of $\Delta(\nu)$ in $\Delta(\lambda) \otimes \Delta(\mu)$. Note that $\mathrm{LR}^\nu_{\lambda\mu} = 0$ unless $\pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)$, the equations (10.1.6) imply $d_\nu = \mathrm{LR}^\nu_{\lambda\mu}$ if $\pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)$ and $d_{\nu,\boldsymbol{\varphi}} = 0$ if $\pi_{\mathbf{m}}(\nu) < \pi_{\mathbf{m}}(\lambda+\mu)$. Thus, we have

$$[\Delta(\lambda) \otimes \Delta(\mu)] = \sum_{\nu \in \varLambda^+_{n,r}(\mathbf{m})} \mathrm{LR}^\nu_{\lambda\mu} [\Delta(\nu)]. \tag{10.1.7}$$

By (10.1.2), for any $k = 1, 2, \dots, r-1$ and any $t \geq 0$, we have

$$\mathcal{X}^+_{(m_k,k),t} \cdot \Big(\bigoplus_{\substack{\nu \in \varLambda_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)}} (\Delta(\lambda) \otimes \Delta(\mu))_\nu \Big) = 0 \tag{10.1.8}$$

since $\pi_{\mathbf{m}}(\nu + \alpha_{(m_k,k)}) > \pi_{\mathbf{m}}(\nu)$. Then, by (10.1.4) and (10.1.8) together with the relation (L2), we see that

(10.1.9)

$$\begin{aligned} &\{ v \in (\Delta(\lambda) \otimes \Delta(\mu))_\nu \mid \mathcal{X}^+_{(i,k),t} \cdot v \text{ for all } (i,k) \in \varGamma'(\mathbf{m}) \text{ and } t \geq 0 \} \\ &= \{ v \in (\Delta(\lambda) \otimes \Delta(\mu))_\nu \mid e_{(i,k)} \cdot v \text{ for all } (i,k) \in \varGamma(\mathbf{m}) \setminus \{ (m_k, k) \mid 1 \leq k \leq r \} \} \end{aligned}$$

for $\nu \in \varLambda^+_{n,r}(\mathbf{m})$ such that $\pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)$, where $e_{(i,k)} \in U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$ acts on $\Delta(\lambda) \otimes \Delta(\mu)$ through the injection (2.16.2). On the other hand, $\bigoplus_{\substack{\nu \in \varLambda_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)}} (\Delta(\lambda) \otimes \Delta(\mu))_\nu$ is a $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-submodule of $\Delta(\lambda) \otimes \Delta(\mu)$ and we have

(10.1.10)

$$\bigoplus_{\substack{\nu \in \varLambda_{n,r}(\mathbf{m}) \\ \pi_{\mathbf{m}}(\nu) = \pi_{\mathbf{m}}(\lambda+\mu)}} (\Delta(\lambda) \otimes \Delta(\mu))_\nu \cong \bigoplus_{\nu \in \varLambda^+_{n,r}(\mathbf{m})} \mathrm{LR}^\nu_{\lambda\mu} \, \Delta_{\mathfrak{gl}_{m_1}}(\nu^{(1)}) \otimes \cdots \otimes \Delta_{\mathfrak{gl}_{m_r}}(\nu^{(r)})$$

as $U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})$-modules by comparing the character (note [W2, Lemma 2.6]). By (10.1.7), (10.1.9) and (10.1.10), we see that

$$\Delta(\lambda) \otimes \Delta(\mu) \cong \bigoplus_{\nu \in \varLambda^+_{n,r}(\mathbf{m})} \mathrm{LR}^\nu_{\lambda\mu} \, \Delta(\nu)$$

as $U(\mathfrak{g}_{\widetilde{\mathbf{Q}}}(\mathbf{m}))$-modules. □

Remarks 10.2.

(i) For $M, N \in \mathcal{C}_{\widetilde{Q}}(\mathbf{m})$, we see that $\operatorname{ch}(M \otimes N) = \operatorname{ch}(M) \operatorname{ch}(N)$ by definition of characters. Then the decomposition (10.1.1) gives an interpretation of the formula (9.2.1) (Proposition 9.4 (iii)) in the category $\mathcal{C}_{\widetilde{\mathbf{Q}}}(\mathbf{m})$.

(ii) We conjecture that the algebra $\mathcal{U}_{q,\widetilde{\mathbf{Q}}}(\mathbf{m})$ has a structure as a Hopf algebra. Then we also conjecture the similar decomposition for the tensor product of Weyl modules of $\mathscr{S}^{\widetilde{\mathbb{K}}}_{n,r}(\mathbf{m})$ $(n \geq 0)$ as in (10.1.1).

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